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Shumei Hirai **Akio Matsumoto**
Keiko Nakayama **Ferenc Szidarovszky**

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Shumei Hirai† Akio Matsumoto‡

Keiko Nakayama§ Ferenc Szidarovszky¶

Abstract

In this study, we consider how an environmental policy controls NPS (non-point source) pollution when the standard environmental policies cannot be applied since the regulator is unable to observe individual emission levels of NPS pollution. In Cournot competition, the firms determine the optimal output and abatement technology levels to maximize their profits, taking the environmental tax rate as given. It is analytically and numerically demonstrated that the ambient charge tax rate can control the total size of NPS pollution under various circumstances, such as duopoly or oligopoly, with or without product differentiation when the firms are homogeneous or heterogeneous.

Keywords: NPS pollution, Ambient charge, Cournot oligopoly competition, Homogeneity, Heterogeneity.

Contents:

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†Department of Economics, Seinan-Gakuin University, 6-2-92, Nishijin, Sawara, Fukuoka, 814-0002, Japan. hirai@seinan-gu.ac.jp

‡The corresponding author, Department of Economics, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan, akiom@tamacc.chuo-u.ac.jp.

§Department of Economics, Chukyo University, 101-2, Yagoto-Honmachi, Showa, Nagoya, 466-0825, Japan. nakayama@mecl.chukyo-u.ac.jp

¶Department of Mathematics, Corvinus University, Budapest, Fövám tér 8, 1093, Hungary, szidarka@gmail.com.

1 Introduction

Oligopoly models are among the most frequently discussed topics in mathematical economics literature. The classical single-product model without product differentiation has been extended in many different directions, including product differentiation, multiproduct models, labor-managed oligopolies, and rent-seeking models, among many others (Okuguchi, 1976; Okuguchi and Szidarovszky, 1999). Oligopoly models, including environmental issues, became an essential line of research because of their practical importance and theoretical challenges. There are several lines of research in this broad field. In the case of point-source pollutants, the regulator knows each firm's pollution level to punish or reward the firm individually. However, in the case of nonpoint source (NPS, henceforth) pollution, the regulator cannot monitor individual emissions with low cost and sufficient accuracy. Therefore, standard instruments of environmental policy are not possible.

The effects of different environmental regulation policies are examined by several researchers, including Downing and White (1986), Jung et al. (1996) and Montero (2002). Xepapadeas (2011) summarizes the different control methods. According to Segerson (1988), who suggests monitoring the ambient concentration of NPS pollutants, the regulator first selects an environmental standard, imposes a uniform tax on the pollutants if the concentration is above this standard, and gives uniform reward if it is below. The regulator and the firms have two decision variables. The regulator decides about the environmental standard and the environmental tax rate. The firms can select their abatement technologies and output volumes. One important question is determining how the environmental standard and the tax rate affect the emission concentration. Ganguli and Raju (2012) examined Bertrand duopolies and showed that increased ambient charges might increase the emission concentration, called the "perverse" effect. Raju and Ganguli (2013) numerically showed the effectiveness of the ambient charge in Cournot duopolies. This result was shown analytically by Sato (2017). The n-firm generalization of this model was investigated by Matsumoto et al. (2017) in a dynamic framework, and the corresponding Bertrand model was examined by Ishikawa et al. (2019), showing that the sign of the effect depends on the number of firms, the degree of substitutability, and the heterogeneity of the abatement technologies of the firms. Matsumoto et al. (2018b) considered a one-stage and a two-stage Bertrand duopoly, and a three-stage Cournot oligopoly was introduced and examined by Matsumoto et al. (2020), where, in the first stage, the regulator determines the tax rate of the ambient charge to maximize social welfare, in the second stage each firm selects optimal abatement technology. In the third stage, the firms decide on their optimal output levels.

This paper considers a Cournot oligopoly. The regulator determines the ambient charge rate to maximize the random welfare function by taking the environmental standard as given.[1] Each firm maximizes its profit as a bivariable

[1] We will consider determining the optimal tax rate maximizing social welfare in future

function with decision variables being the ambient technology and production level. The profit of each firm includes the revenue, the production cost, the ambient charge (or reward) and the technology installment cost. The equilibrium will be determined, and the effectiveness of the ambient charge is shown in various circumstances, including duopoly or oligopoly with or without product differentiation with homogeneous or heterogeneous firms.

The paper is developed as follows. Section 2 constructs the basis n-firm model in which the firms make the optimal choices of output and abatement technology. Section 3 first assumes that the firms are homogeneous and then shows that the ambient charge effectively controls the NPS pollution in duopoly and oligopoly markets. Section 4 replaces the homogeneous with heterogeneous assumptions and considers the ambient charge effect in a duopoly with and without product differentiation. Section 6 offers concluding remarks and outlines further research directions.

2 n-Firm Oligopoly Model

We recapitulate the main structure of the general oligopoly model constructed by Matsumoto et al. (2017). The linear price (i.e., inverse demand) function of good k is

$$p_k = \alpha_k - q_k - \gamma \sum_{j \neq k}^{n} q_j \text{ for } k = 1, 2, ..., n, \tag{1}$$

in which q_k is the quantity of good k, p_k is its price, γ is the substitution parameter measuring the degree of differentiation between the goods, and α_k denotes the quality of good k.[2] In this study, we assume that $0 < \gamma \leq 1$ to confine our analysis to the case in which the goods are substitutes. It is further assumed that the production cost function of firm k is linear and there is no fixed cost. $c_k > 0$ denotes the marginal production cost. To avoid negative optimal production, we impose the traditional assumption that $\alpha_k - c_k$ is positive. We can call this difference the *market size* of firm k and denote it by β_k. Each firm produces output and emits pollution. It is assumed that one unit of production emits one unit of pollution. Let ϕ_k denote the pollution abatement technology of firm k ($0 \leq \phi_k \leq 1$) with a pollution-free technology if $\phi_k = 0$ and a fully-discharged technology if $\phi_k = 1$. If firm k believes that the competitors' outputs will remain unchanged, then its profit is

$$\pi_k(q_k, \phi_k) = \left(\beta_k - q_k - \gamma \sum_{j \neq k}^{n} q_j \right) q_k - (1 - \phi_k)^2 - \theta \left(\sum_{j \neq k}^{n} \phi_j q_j - \bar{R} \right) \tag{2}$$

studies.

[2]The price functions in (1) can be derived as solutions of the utility maximization of the following form,

$$U(\boldsymbol{q}) = \sum_{i=1}^{n} \alpha_i q_i - \frac{1}{2} \left(\sum_{i=1}^{n} q_i^2 + 2\gamma \sum_{i=1}^{n} q_i \sum_{j \neq i}^{n} q_j \right) - \sum_{i=1}^{n} p_i q_i.$$

Here, α_i is a proxy for the quality of good i because an increase in α_i positively affects the utility level.

where $\bar{R}$ is the ambient standard set by a regulator, θ is the ambient tax rate and $(1-\phi_i)^2$ is the installation cost of technology. The rate θ is measured in some monetary unit per emission. It is positive and can be larger than unity (e.g., dollar/ton, euro/kg, etc).

Firm k strategically selects optimal output and abatement technology levels, q_k and ϕ_k, to maximize its profit. Differentiating (2) with respect to q_k and ϕ_k presents the first-order conditions for interior maxima,

$$\frac{\partial \pi_k}{\partial q_k} = \beta_k - 2q_k - \gamma Q_{-k} - \theta\phi_k = 0 \tag{3}$$

and

$$\frac{\partial \pi_k}{\partial \phi_k} = -\theta q_k + 2\left(1-\phi_k\right) = 0. \tag{4}$$

where $Q_{-k} = \sum_{j \neq k} q_j$ is the output of the rest of the industry. The second-order conditions are

$$\frac{\partial^2 \pi_k}{\partial q_k^2} = -2 < 0, \ \frac{\partial^2 \pi_k}{\partial \phi_k^2} = -2 < 0, \ \frac{\partial^2 \pi_k}{\partial q_k^2}\frac{\partial^2 \pi_k}{\partial \phi_k^2} - \left(\frac{\partial^2 \pi_k}{\partial q_k \partial \phi_k}\right)^2 = 4 - \theta^2 > 0$$

where the last inequality holds if $\theta < 2$. Using (4), we rewrite the first-order conditions (3) for the optimal output as

$$\left(4-\theta^2\right) q_k + 2\gamma Q_{-k} = 2\left(\beta_k - \theta\right) \text{ for } k = 1, 2, ..., n$$

or in a vector form,

$$\boldsymbol{B}\boldsymbol{q} = \boldsymbol{A} \tag{5}$$

where

$$\boldsymbol{q} = \left(q_k\right)_{(n,1)}, \ \boldsymbol{A} = \left(2\left(\beta_k - \theta\right)\right)_{(n,1)}$$

and

$$\boldsymbol{B} = \left(B_{ij}\right)_{(n,n)} \text{ with } B_{ii} = 4-\theta^2 \text{ and } B_{ij} = 2\gamma \text{ for } i \neq j.$$

Since $\boldsymbol{B}$ is invertible, solving (5) yields the Cournot outputs,

$$\boldsymbol{q} = \boldsymbol{B}^{-1}\boldsymbol{A} \tag{6}$$

where the diagonal and off-diagonal elements of $\boldsymbol{B}^{-1}$ are, respectively,

$$\frac{4-\theta^2+2\left(n-2\right)\gamma}{\left(4-\theta^2+2\left(n-1\right)\gamma\right)\left(4-\theta^2-2\gamma\right)} \text{ and } \frac{-2\gamma}{\left(4-\theta^2+2\left(n-1\right)\gamma\right)\left(4-\theta^2-2\gamma\right)}.$$

To guarantee $4-\theta^2-2\gamma = (2-\theta^2)+2(1-\gamma) > 0$ for analytical simplicity, we impose the following under which the denominators of the above elements are positive, and the second-order conditions are fulfilled:

Assumption 1. $\theta < \sqrt{2}$

The Cournot equilibrium output of firm k is

$$q_k^C = \frac{2\left[\left(4-\theta^2+2\left(n-2\right)\gamma\right)\beta_k - 2\gamma\beta_{-k} - \theta\left(4-\theta^2-2\gamma\right)\right]}{\left(4-\theta^2+2\left(n-1\right)\gamma\right)\left(4-\theta^2-2\gamma\right)} \tag{7}$$

where we introduce a new notation, $\beta_{-k} = \sum_{j\neq k}^{n}\beta_j$. From (4) and (7), we also obtain the optimal abatement technology of firm k,

$$\phi_k^C = 1 - \frac{\theta}{2}q_k^C. \tag{8}$$

The right-hand side of equation (8) with (7) is expressed in a form that will facilitate later calculations,

$$\frac{\theta\left[2\gamma\beta_{-k} - \left(4-\theta^2+2\left(n-2\right)\gamma\right)\beta_k\right] + 2(2+(n-1)\gamma)\left(4-\theta^2-2\gamma\right)}{\left(4-\theta^2+2\left(n-1\right)\gamma\right)\left(4-\theta^2-2\gamma\right)}. \tag{9}$$

Solving (8) for q_k^C yields an simplified form,

$$q_k^C = \left(1-\phi_k^C\right)\frac{2}{\theta}.$$

The Cournot output is non-negative if $\phi_k^C \leq 1$ and not greater than the upper bound, $2/\theta$, if $\phi_k^C \geq 0$.

We will search for the parametric condition under which the optimal level of the abatement technology is positive and not greater than unity. With (9), solving $\phi_k^C = 0$ and $\phi_k^C = 1$ for β_{-k} presents

$$\beta_{-k} = f_0\left(\beta_k\right) \equiv \frac{4-\theta^2+2\gamma(n-2)}{2\gamma}\beta_k - \frac{(2+(n-1)\gamma)\left(4-\theta^2-2\gamma\right)}{\gamma\theta} \tag{10}$$

and

$$\beta_{-k} = f_1\left(\beta_k\right) \equiv \frac{4-\theta^2+2\gamma(n-2)}{2\gamma}\beta_k - \frac{\theta\left(4-\theta^2+2\gamma\right)}{2\gamma}. \tag{11}$$

These equations are developed as an n-dimensional simultaneous system of linear equations in β_k

$$\beta_{-k} - A\beta_k = B \text{ for } k = 1, 2, ...n \tag{12}$$

and

$$\beta_{-k} - A\beta_k = C \text{ for } k = 1, 2, ...n \tag{13}$$

where

$$A = \frac{4-\theta^2+2\gamma(n-2)}{2\gamma},\ B = -\frac{(2+(n-1)\gamma)\left(4-\theta^2-2\gamma\right)}{\gamma\theta} \text{ and } C = -\frac{\theta\left(4-\theta^2+2\gamma\right)}{2\gamma}.$$

Solving, respectively, (12) and (13) for β_k yields the maximum and minimum denoted as β_M and β_m,

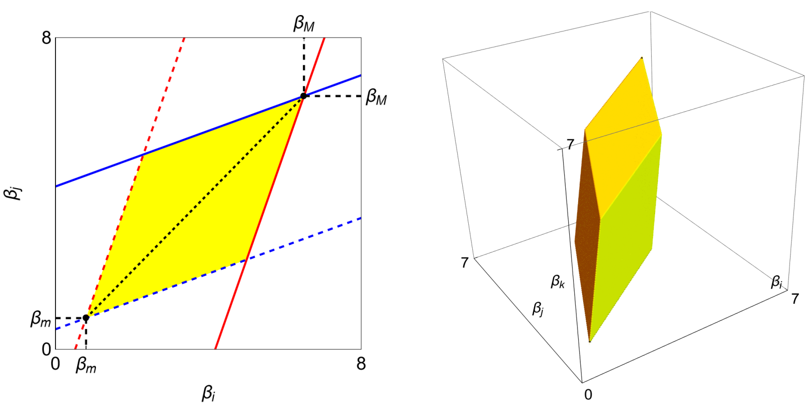
$$\beta_M = \frac{2\left[2+(n-1)\gamma\right]}{\theta} \text{ and } \beta_m = \theta. \tag{14}$$

Since, equations (10) and (12) are alternative forms of $\phi_k^C = 0$, and equations (11) and (12) are alternative forms of $\phi_k^C = 1$, conditions, $\phi_k^C = 0$ or $\phi_k^C = 1$ holds if $\beta_k = \beta_M$ or $\beta_k = \beta_m$ for $k = 1, 2, ..., n$. Since $\beta_M > \beta_m$, $0 \leq \phi_k^C \leq 1$ holds if $\beta_m \leq \beta_k \leq \beta_M$. We summarize the feasible conditions for the optimal solutions as follows:

Theorem 1 *The optimal productions and optimal abatement technologies satisfy the feasible conditions,* $0 \leq \phi_k^C \leq 1$ *and* $0 \leq q_k^C \leq \theta/2$*, if the market sizes are in the set,*

$$M_n = \{(\beta_1, \beta_2, ...\beta_n) \mid f_1(\beta_k) \leq \beta_{-k} \leq f_0(\beta_k) \text{ and } \beta_m \leq \beta_k \leq \beta_M\} \text{ for } k = 0, 1, 2, ..., n$$

In the case of duopoly (i.e., $n = 2$), the set M_2 is the diamond-shaped yellow region in Figure 1(A) surrounded by the solid red and dotted-red lines of $\phi_i^C = 0$ and $\phi_i^C = 1$ and by the solid blue and dotted-blue lines of $\phi_j^C = 0$ and $\phi_j^C = 1$. Notice that the solid blue and red curves intersect at $\beta_i = \beta_j = \beta_M$ and so do the dotted blue and red curves at $\beta_i = \beta_j = \beta_m$. If the two firms are homogeneous (i.e., $\beta_i = \beta_j$), then the feasible region is shrunk to the dotted diagonal between β_m and β_M. In the case of triopoly (i.e.,, $n = 3$), the feasible region M_3 is described by the hexahedron with diamond-shaped faces as seen in Figure 1(B).



(A) $n = 2$ (B) $n = 3$

Figure 1. Feasible regions with $\theta = 4/5$ and $\gamma = 3/5$

From (7) and (8), we have the following,

$$q_k^C - \bar{q}^C = \frac{2}{4 - \theta^2 - 2\gamma}(\beta_k - \bar{\beta})$$

and

$$\phi_k^C - \bar{\phi}^C = -\frac{\theta}{2}\left(q_k^C - \bar{q}^C\right)$$

where the corresponding averages are defined as

$$\bar{\beta} = \frac{1}{n}\sum_{j=1}^{n}\beta_j,\ \bar{q}^C = \frac{1}{n}\sum_{j=1}^{n}q_j^C \text{ and } \bar{\phi}^C = \frac{1}{n}\sum_{j=1}^{n}\phi_j^C.$$

The following results concerning the optimal decisions among the firms are clear.

Theorem 2 *The firm with a larger market size than the average produces more output and adopts more efficient abatement technology than the corresponding averages,*

$$\beta_k \gtreqless \bar{\beta} \textit{ implies } q_k^C \gtreqless \bar{q}^C \textit{ and } \phi_k^C \lesseqgtr \bar{\phi}^C.$$

3 Homogeneous Firms

To examine the effects caused by a change in the ambient charge rate on the optimal solutions of output and technology and on the total pollution, we start with the simpler case in which the firms are homogeneous. For this purpose, we impose the following:

Assumption 2. $\beta_i = \beta$ for $i = 1, 2, ..., n$.

Depending on the selected parametric values of γ and n, we still have four exclusive cases,

	$n = 2$	$n \geq 3$
$\gamma = 1$	Ho_1	Ho_3
$\gamma < 1$	Ho_2	Ho_4

where the firms are homogeneous and

(1) duopoly without product differentiation in case Ho_1:

(2) duopoly with product differentiation in case Ho_2;

(3) oligopoly without product differentiation in case Ho_3;

(4) oligopoly with product differentiation in case Ho_4.

Accordingly, the remaining part of this section is divided into two subsections: in the first subsection, cases Ho_2 and Ho_1 are mainly considered, while case Ho_3 is focused on and case Ho_4 is briefly mentioned in the second one.

3.1 Duopoly Firms with Product Differentiation

We first examine case Ho_2 in which the number of the homogeneous firms is limited to 2 and the goods are differentiated:

Assumption 3. $n = 2$ and $\gamma < 1$

A direct consequence of Assumption 2 is that their optimal decisions are identical. In particular, (9) with $n = 2$ give identical optimal abetment technology,

$$\phi_{II}^{C} = \frac{\theta\left(\beta_M - \beta\right)}{4 - \theta^2 + 2\gamma}. \tag{15}$$

Here, the subscript II means case Ho_2. From (15),

$$1 - \phi_{II}^{C} = \frac{\theta\left(\beta - \beta_m\right)}{4 - \theta^2 + 2\gamma}.$$

where, from (14) with $n = 2$,

$$\beta_M(\theta) = \frac{2(2+\gamma)}{\theta} \text{ and } \beta_m(\theta) = \theta.$$

Optimal technology ϕ_{II}^{C} is then substituted into (8) to have the optimal output,

$$q_{II}^{C} = \frac{2\left(1 - \phi_{II}^{C}\right)}{\theta} = \frac{2\left(\beta - \beta_m\right)}{4 - \theta^2 + 2\gamma} > 0 \text{ if } \phi_{II}^{C} < 1. \tag{16}$$

Since the denominator of ϕ_{II}^{C} in (15) is positive, we have the following as a Corollary of Theorem 1:

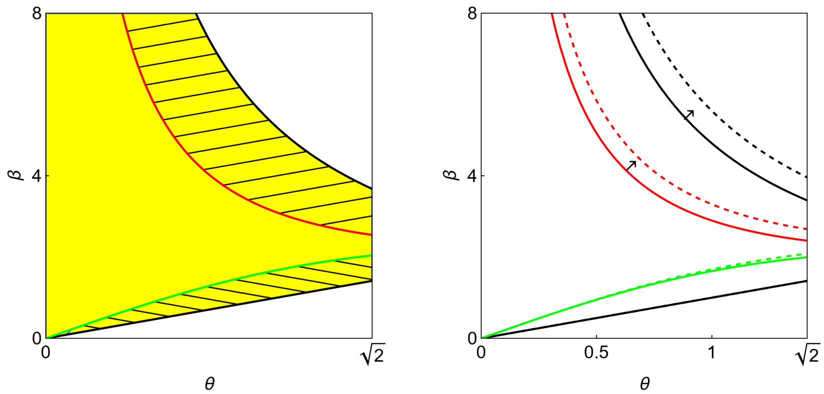
Corollary 1 *Under Assumptions 1, 2 and 3, the optimal output and optimal technology satisfy* $0 \leq q_{II}^{C} \leq 2/\theta$ *and* $0 \leq \phi_{II}^{C} \leq 1$ *if and only if* $\beta_m \leq \beta \leq \beta_M$.

Notice that Figure 2(A) visualizes Corollary 1. The negative-sloping black locus of $\beta = \beta_M(\theta)$ and the positive-sloping black locus of $\beta = \beta_m(\theta)$ are the upper and lower boundaries of the yellow region.[3] The feasible conditions of $0 \leq \phi_{II}^{C} \leq 1$ and $0 \leq q_{II}^{C} \leq 2/\theta$ hold in the yellow region. Figure 2(B) illustrates the effects of an increase of γ from $2/5$ to $4/5$ on the locations of various curves, indicating that the greater the substitutability, the greater the stability region:

(i) an enlargement effect by shifting the upper black, green and red curve upwards;

[3] We will refer to the red negative- and blue positive-sloping curves inside the yellow region of Figure 2(A) soon after.

(ii) no effect on the lower black curve that depends only on θ.



(A) $\gamma = 3/5$ (B) $\gamma = 2/5$ to $\gamma = 4/5$

Figure 2. Feasible region of (θ, β) for $0 < \phi_{II}^{C} < 1$ and $\theta = 4/5$

The total amount of production pollution is the sum of the individual pollution. In the homogeneous duopoly case, it is given by

$$E_{II}^{C}(\theta) = 2\phi_{II}^{C}(\theta) q_{II}^{C}(\theta).$$

Differentiating $\phi_{II}^{C}(\theta)q_{II}^{C}(\theta)$ with respect to θ yields, after arranging the terms,

$$-\frac{2\left[(4+2\gamma+3\theta^2)\beta^2 - 2\theta(12+6\gamma+\theta^2)\beta + 2(2+\gamma)(2+\gamma+3\theta^2)\right]}{(4+2\gamma-\theta^2)^3}. \tag{17}$$

The denominator is positive. The bracketed terms of the numerator form a quadratic polynomial in β, having a positive constant term. Its discriminant is negative,

$$D/4 = -\left(4-\theta^2+2\gamma\right)^3 < 0.$$

Hence, the polynomial is positive for any β, implying that the derivative in (17) is negative. Therefore,

$$\frac{dE_{II}^{C}(\theta)}{d\theta} = 2\frac{d\left[\phi_{II}^{C}(\theta)q_{II}^{C}(\theta)\right]}{d\theta} < 0. \tag{18}$$

The direction of inequality means that an increase of the ambient charge rate decreases the total amount of pollution. This result is summarized as follows:

Theorem 3 *In a homogeneous duopoly with product differentiation, the ambient charge rate can control the concentration of the NPS pollutions,*

$$\frac{dE_{II}^{C}}{d\theta} < 0.$$

We consider why the ambient charge can be effective in controlling pollution. Differentiating (15) and (16) with respect to θ gives

$$\frac{d\phi_{II}^C}{d\theta} = \frac{4+2\gamma+\theta^2}{4+2\gamma-\theta^2}\left[\frac{2\theta(4+2\gamma)}{4+2\gamma+\theta^2}-\beta\right] \tag{19}$$

and

$$\frac{dq_{II}^C}{d\theta} = \frac{4\theta}{\left(4+2\gamma-\theta^2\right)^2}\left[\beta-\frac{4+2\gamma+\theta^2}{2\theta}\right]. \tag{20}$$

Accordingly, two new functions are introduced,

$$\beta_0(\theta) = \frac{2\theta(4+2\gamma)}{4+2\gamma+\theta^2} \text{ and } \beta_1(\theta) = \frac{4+2\gamma+\theta^2}{2\theta}.$$

The $\beta = \beta_0(\theta)$ curve corresponds to the positive-sloping green curve located just above the $\beta = \beta_m(\theta)$ line in Figure 2(A). Equation (19) leads to the following relations,

$$\frac{d\phi_{II}^C}{d\theta} \gtreqless 0 \text{ according to } \beta \lesseqgtr \beta_0(\theta).$$

Thus in the shaded yellow region surrounded by the two curves, $\beta = \beta_m(\theta)$ and $\beta = \beta_0(\theta)$, the following inequality holds:

$$\frac{d\phi_{II}^C}{d\theta} > 0.$$

Individual firms paradoxically respond to government's increased ambient charge taxation by adopting less abatement technologies when the market sizes are small.

Similarly, the $\beta = \beta_1(\theta)$ curve corresponds to the negative-sloping red curve located just below the $\beta = \beta_M(\theta)$ curve where the following relations hold:

$$\frac{dq_{II}^C}{d\theta} \gtreqless 0 \text{ according to } \beta \gtreqless \beta_1(\theta).$$

In the shaded yellow region surrounded by these two curves, $\beta = \beta_M(\theta)$ and $\beta = \beta_1(\theta)$, we have

$$\frac{dq_{II}^C}{d\theta} > 0.$$

The firms increase their optimal production in response to higher taxes when their market sizes are larger. Summarizing the results, we can divide the stable yellow region into three subregions in which

$$\begin{aligned}
&(i)\ \frac{dq_{II}^C}{d\theta} < 0 \text{ and } \frac{d\phi_{II}^C}{d\theta} \geq 0 \text{ for } \beta_m(\theta) \leq \beta \leq \beta_0(\theta),\\
&(ii)\ \frac{dq_{II}^C}{d\theta} < 0 \text{ and } \frac{d\phi_{II}^C}{d\theta} < 0 \text{ for } \beta_0(\theta) < \beta < \beta_1(\theta),\\
&(iii)\ \frac{dq_{II}^C}{d\theta} \geq 0 \text{ and } \frac{d\phi_{II}^C}{d\theta} < 0 \text{ for } \beta_1(\theta) \leq \beta \leq \beta_M(\theta).
\end{aligned} \tag{21}$$

If the ambient tax rate change has an unfavorable effect on the variables, we call it a *perverse* effect. Although the regulator cannot observe the individual firm's reactions to a change in θ, their optimal responses are summarized as follows:

Theorem 4 *The ambient tax rate has a perverse effect on the optimal technology if β is small enough in the sense that $\beta_m(\theta) \leq \beta \leq \beta_0(\theta)$ and on the optimal output if β is large enough in the sense that $\beta_1(\theta) \leq \beta \leq \beta_M(\theta)$ whereas it has a normal effect on both variables if β takes a normal value in the sense that $\beta_0(\theta) < \beta < \beta_1(\theta)$.*

We now turn attention to the relative magnitude of these normal and perverse effects. Differentiating $E_{II}^C(\theta) = 2\phi_{II}^C(\theta)q_{II}^C(\theta)$ with respect to θ and arranging the terms present

$$\frac{dE_{II}^C(\theta)}{d\theta} = 2\frac{\phi_{II}^C(\theta)q_{II}^C(\theta)}{\theta}\left(\varepsilon_\phi^C + \varepsilon_q^C\right)$$

where ε_ϕ^C and ε_q^C denote the elasticities of technology and output with respect to ambient charge rate defined as

$$\varepsilon_\phi^C = \frac{\theta}{\phi_{II}^C}\frac{d\phi_{II}^C}{d\theta} \text{ and } \varepsilon_q^C = \frac{\theta}{q_{II}^C}\frac{dq_{II}^C}{d\theta}$$

Since θ can have a perverse effect, as discussed, the sign of $dE_{II}^C(\theta)/d\theta$ depends on the relative magnitudes between ε_ϕ^C and ε_q^C in absolute values. In case (ii) of (21) or in the non-striped yellow region of Figure 2(A), both elasticities are negative, implying that $dE_{II}^C(\theta)/d\theta < 0$. On the other hand, the sign of $dE_{II}^C(\theta)/d\theta$ in case (i) or in case (iii) seems to be ambiguous because the two elasticities are of opposite signs. However, Theorem 3 has already confirmed $dE_{II}^C(\theta)/d\theta < 0$ in both regions: in case (i), the negative elasticity of the optimal output dominates the positive elasticity of the technology; in case (iii), the negative elasticity of the optimal technology dominates the positive elasticity of the optimal output. Hence, in both cases, the normal effect dominates the perverse effect. This is summarized as a Corollary of Theorem 3.

Corollary 2 *Although a change in the ambient tax rate possibly induces the opposite-signed normal and perverse effects, depending on the value of β, the normal effect always dominates the perverse effect,*

$$\varepsilon_\phi^C + \varepsilon_q^C < 0 \textit{ for } \beta_m(\theta) \leq \beta \leq \beta_0(\theta) \textit{ or for } \beta_1(\theta) \leq \beta \leq \beta_M(\theta).$$

Summarizing the results obtained in case Ho_2, we see that the substitution parameter γ has the enlargement effect and the ambient tax rate θ can control the NPS pollution. It is not difficult to verify the same results obtained in Theorem 3 and Corollaries 1 and 3 in case Ho_1, for the more simplified case where $n = 2$ and $\gamma = 1$.

3.2 Oligopoly without Product Differentiation

In this section, we will confine ourselves to the general n-oligopoly case, Ho_3, in which we get rid of Assumption 3 and impose, only for analytical simplicity, the assumption that the goods are not differentiated.

Assumption 4. $n > 2$ and $\gamma = 1$.

The homogenous assumption (i.e., $\beta_i = \beta$) and no product differentiation (i.e., $\gamma = 1$) simplify (7) and (9),

$$q_{III}^C = \frac{2(\beta - \theta)}{2(n+1) - \theta^2}, \tag{22}$$

$$\phi_{III}^C = \frac{2(n+1) - \beta\theta}{2(n+1) - \theta^2} \tag{23}$$

and

$$1 - \phi_{III}^C = \frac{\theta(\beta - \theta)}{2(n+1) - \theta^2} \tag{24}$$

where the subscript III means case Ho_3. From (14) and Assumption 4, $0 \leq \phi_{III}^C \leq 1$ and $0 \leq q_{III}^C \leq 1$ hold if

$$\theta \leq \beta \leq \frac{2(n+1)}{\theta}. \tag{25}$$

We now consider the effects of changing θ on these optimal values. Differentiating ϕ_{III}^C and q_{III}^C with respect to θ yields the following:

$$\frac{\partial \phi_{III}^C}{\partial \theta} = \frac{2(n+1) + \theta^2}{\left[2(n+1) - \theta^2\right]^2} \left[\beta_0(\theta, n) - \beta\right]$$

and

$$\frac{\partial q_{III}^C}{\partial \theta} = \frac{4\theta}{\left[2(n+1) - \theta^2\right]^2} \left[\beta - \beta_1(\theta, n)\right]$$

where

$$\beta_0(\theta, n) = \frac{4\theta(n+1)}{2(n+1) + \theta^2} \text{ and } \beta_1(\theta, n) = \frac{2(n+1) + \theta^2}{2\theta}.$$

In Figure 3(A), the yellow region is surrounded by the two black curves obtained in (25), the positive-sloping $\beta = \theta$ curve and the negative-sloping $\beta = 2(n+1)/\theta$ curve. The loci of $\beta = \beta_0(\theta, n)$ and $\beta = \beta_1(\theta, n)$ are the green and red curves. So we have

$$\frac{\partial \phi_{III}^C}{\partial \theta} \gtreqless 0 \text{ according to } \beta \lesseqgtr \beta_0(\theta, n)$$

and

$$\frac{\partial q_{III}^C}{\partial \theta} \gtreqless 0 \text{ according to } \beta \gtreqless \beta_1(\theta, n).$$

In the upper shaded yellow region of Figure 3(A) between the negative-sloping black and red curves, the optimal production has a perverse effect,

$$\frac{\partial \phi^C_{III}}{\partial \theta} < 0 \text{ and } \frac{\partial q^C_{III}}{\partial \theta} > 0.$$

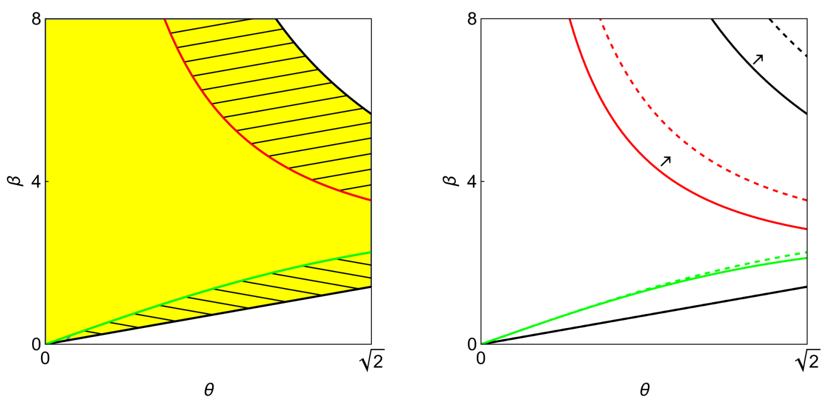
On the other hand, in the lower shaded yellow region between the positive-sloping black and green curves, the optimal abatement technology has a perverse effect,

$$\frac{\partial \phi^C_{III}}{\partial \theta} > 0 \text{ and } \frac{\partial q^C_{III}}{\partial \theta} < 0.$$

In the yellow region between the red and green curves, both have the normal effect,

$$\frac{\partial \phi^C_{III}}{\partial \theta} < 0 \text{ and } \frac{\partial q^C_{III}}{\partial \theta} < 0.$$

Figure 3(B) depicts the effects of increasing the value of n from 3 to 4 on the optimal values, which shift the upper black, red and green curves upward. Hence, the larger n enlarges the feasible yellow region. Figures 3(A) and (B) with $n = 3$ and $\gamma = 1$ look quite similar to Figures 2(A) and (B) with $n = 2$ and $\gamma < 1$. The effects caused by changing the substitution term have the similar effect caused by changing the number of firms. One quantitative difference is that β_M in case Ho_2 has the upper bound in γ (i.e., $6/\theta$) whereas β_M in case Ho_3 has no upper bound in n.



(A) $n = 3$ and $\theta = 4/5$ (B) $n = 3$ and $n = 4$

Figure 3. Feasible region and effects caused by increasing n

The total amount of production pollution is the sum of individual pollution,

$$E^C_{III}(\theta) = n\phi^C_{III}(\theta)q^C_{III}(\theta).$$

(22) and (23) reduce it to a simpler and manageable form

$$E^C_{III}(\theta) = n\frac{2(\beta - \theta)[2(n+1) - \beta\theta]}{[2(n+1) - \theta]^2}. \tag{26}$$

Differentiating (26) with respect to θ yields

$$\frac{dE^C_{III}(\theta)}{d\theta} = -2n\frac{s(n,\theta)\beta^2 - 2u(n,\theta)\beta + v(n,\theta)}{\left[2(n+1)-\theta^2\right]^3} \tag{27}$$

with

$$\begin{aligned}
s(n,\theta) &= 2(n+1) + 3\theta^2 > 0, \\
u(n,\theta) &= \theta\left[6(n+1) + \theta^2\right] > 0, \\
v(n,\theta) &= 2(n+1)\left[2(n+1) + 3\theta^2\right] > 0.
\end{aligned}$$

The denominator of (27) is positive under Assumption 1. The numerator is a quadratic polynomial in β, having the positive constant term, $v(n,\theta) > 0$. Its discriminant is negative,

$$\frac{D}{4} = -\left[2(n+1)-\theta^2\right]^3 < 0.$$

As a result, the numerator is positive for any $\beta > 0$. Therefore, we arrive at the following:

Theorem 5 *In a homogeneous oligopoly market without production differentiation, the ambient charge tax rate is effective in controlling the concentration of the NPS pollution,*

$$\frac{dE^C_{III}(\theta)}{d\theta} < 0.$$

Applying the procedure developed in case Ho_3 for case Ho_4, we can probably see that (i) the substitution parameter γ has an enlargement effect on the feasible region, and the ambient charge has pollution controllability. The homogeneity assumption contributes to simplifying the analysis.

4 Heterogeneous Firms

In this section, we consider the ambient charge on the optimal behavior and the total pollution when the firms are heterogeneous. To this end, we replace the homogeneous assumption (i.e., Assumption 2) with the heterogeneous assumption,

Assumption 5 $\beta_j \neq \beta_j$ for $j = 1, 2, ..., n$ and $i \neq j$.

We then study how the ambient charge affects the heterogeneous firms in the four exclusive cases, depending on the selected parameter values:

	$n = 2$	$n \geq 3$
$\gamma = 1$	He_1	He_3
$\gamma < 1$	He_2	He_4

where the heterogeneous firms form

(1) duopoly without product differentiation in case He_1;

(2) duopoly with product differentiation in case He_2;

(3) oligopoly without product differentiation in case He_3;

(4) oligopoly with product differentiation in case He_4.

Accordingly, this section is divided into two subsections. We consider case He_2 in the first subsection and case He_3 in the second. Cases He_1 and He_4 are briefly mentioned at the end of these subsections.

4.1 Duopoly Firms without Product Differentiation

We start with the simpler case of He_1 in which we impose the following:

Assumption 6. $n = 2$ and $\gamma = 1$

From (7) and (9), the optimal decisions are

$$q_i^I(\theta) = \frac{2\left[(4-\theta^2)\beta_i - 2\beta_j\right] - 2\theta(2-\theta^2)}{(2-\theta^2)\left(6-\theta^2\right)}, \tag{28}$$

$$\phi_i^I(\theta) = \frac{\theta\left[2\beta_j - (4-\theta^2)\beta_i\right] + 6(2-\theta^2)}{(2-\theta^2)\left(6-\theta^2\right)}. \tag{29}$$

Equations (10) and (11) with Assumption 6 determine the boundaries of the feasible region of each firm. For firm i, the locus of $\phi_i^I(\theta) = 0$ is described by

$$\beta_j = f_0(\beta_i) \equiv \frac{4-\theta^2}{2}\beta_i - \frac{3(2-\theta^2)}{\theta}$$

and the locus of $\phi_i^I(\theta) = 1$ by

$$\beta_j = f_1(\beta_i) \equiv \frac{4-\theta^2}{2}\beta_i - \frac{\theta(2-\theta^2)}{2}.$$

In the same way, we obtain the boundaries for firm j, $\beta_i = f_0(\beta_j)$ and $\beta_i = f_1(\beta_j)$. For graphical convenience, solving these equations for β_j to have the boundaries for firm j in terms of β_i,

$$\beta_j = g_0(\beta_i) \equiv \frac{2}{4-\theta^2}\beta_i + \frac{6(2-\theta^2)}{\theta\left(4-\theta^2\right)}$$

and

$$\beta_j = g_1(\beta_i) \equiv \frac{2}{4-\theta^2}\beta_i + \frac{\theta(2-\theta^2)}{4-\theta^2}.$$

As in Figure 1, the lines of $\beta_j = g_k(\beta_i)$ for $k = 0, 1$ construct the upper and lower sides of the diamond-shaped yellow region while the lines of $\beta_j = f_k(\beta_i)$ for

$k = 0, 1$ the left and right sides. The feasible conditions, $0 \leq \phi_i^I(\theta) \leq 1$ and $0 \leq q_i^I(\theta) \leq \theta/2$ hold in the yellow region.

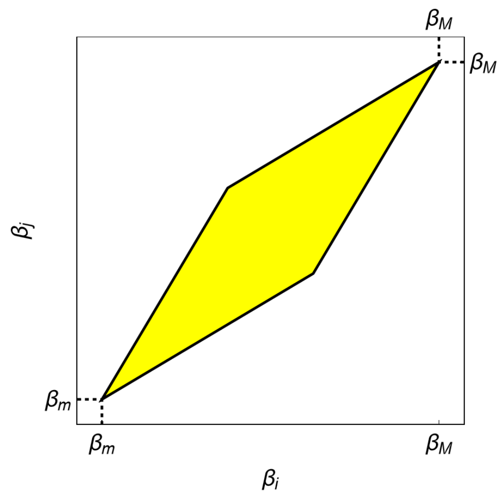


Figure 4. The feasible region with $\theta = 4/5$

The total amount of NPS pollution is the sum of individual pollutions,

$$\boldsymbol{E}_I^C(\theta) = E_i^I(\theta) + E_j^I(\theta) \text{ where } E_k^I(\theta) = \phi_k^I(\theta) q_k^I(\theta) \text{ for } k = i, j.$$

Differentiating it with respect to θ yields

$$\frac{d\boldsymbol{E}_I^C(\theta)}{d\theta} = -2 \frac{F_I\left(\beta_i, \beta_j, \theta\right)}{\left(2-\theta^2\right)^3\left(6-\theta^2\right)^3} \tag{30}$$

where the numerator is written as

$$F_I\left(\beta_i, \beta_j, \theta\right) = A\left(\theta\right)\left(\beta_i^2 + \beta_j^2\right) + B\left(\theta\right)\left(\beta_i + \beta_j\right) + C\left(\theta\right)\beta_i\beta_j + K\left(\theta\right) \tag{31}$$

with

$$A\left(\theta\right) = 240 + 192\theta^2 - 144\theta^4 + 32\theta^6 - 3\theta^8,$$

$$B\left(\theta\right) = -2\theta\left(2-\theta^2\right)^3\left(18+\theta^2\right) < 0,$$

$$C\left(\theta\right) = -8\left(48 + 60\theta^2 - 36\theta^4 + 5\theta^6\right),$$

$$K\left(\theta\right) = 36\left(2+\theta^2\right)\left(2-\theta^2\right)^3 > 0.$$

For $\theta \in (0, \sqrt{2})$, we analytically verify that the denominator of (30) is positive, $B\left(\theta\right) < 0$ and $K\left(\theta\right) > 0$ and numerically check that $A\left(\theta\right) > 0$ and $C\left(\theta\right) < 0$.

We eliminate θ in $F_I\left(\beta_i, \beta_j, \theta\right)$ for the notational simplicity. Notice that $F_I\left(\beta_i, \beta_j\right)$ can be rewritten as a quadratic polynomial in β_i,

$$F_I\left(\beta_i, \beta_j\right) = A_i \beta_i^2 + B_i \beta_i + G_I(\beta_j) \tag{32}$$

where

$$A_i = A\left(\theta\right), \; B_i = B\left(\theta\right) + C\left(\theta\right)\beta_j$$

and

$$G_I(\beta_j) = A\left(\theta\right)\beta_j^2 + B\left(\theta\right)\beta_j + K\left(\theta\right). \tag{33}$$

Notice that $G_I(\beta_j)$ is also a quadratic polynomial in β_j. In the following, we will first focus on $G_I(\beta_j)$ and check its discriminant to find whether $G_I(\beta_j) < 0$ for all $\beta_j \geq 0$. Then, we turn attention to the discriminant of $F_I\left(\beta_i, \beta_j\right)$ in β_i and validate that this discriminant is also negative for all $\beta_i \geq 0$ to show $F_I\left(\beta_i, \beta_j\right) > 0$ for any nonnegative β_i and β_j.

Our first result is the following:

Lemma 1 $G_I(\beta_j) > 0$ *for all* $\beta_j \geq 0$.

Proof. The discriminant of $G_I(\beta_j)$ from (33) is

$$\begin{aligned} \boldsymbol{D}_G &= B\left(\theta\right)^2 - 4A\left(\theta\right)K\left(\theta\right) \\ &= -\theta^2\left(2-\theta^2\right)^3\left(11232 + 144\theta^2 - 4400\theta^4 + 1056\theta^6\right). \end{aligned}$$

The third factor in the last line is positive for $\theta \in (0, \sqrt{2})$; we then have $\boldsymbol{D}_G < 0$. With $A\left(\theta\right) > 0$, $G_I(\beta_j)$ is convex and $G_I(0) = K\left(\theta\right) > 0$. Hence, $G_I(\beta_j) > 0$ for all $\beta_j \geq 0$. ■

With Lemma 1, we have $G_I(\beta_j) > 0$ and $A_i > 0$. Then, $F_I\left(\beta_i, \beta_j\right)$ is convex in β_i and $F_I\left(0, \beta_j\right) = G_I(\beta_j) > 0$. The discriminant of $F_I\left(\beta_i, \beta_j\right)$ with respect to β_i is quadratic in β_j,

$$\begin{aligned} \boldsymbol{D}_F &= B_i^2 - 4A_i G_I(\beta_j) \\ &= \left(C^2 - 4A^2\right)\beta_j^2 + 2B\left(C - 2A\right)\beta_j + \left(B^2 - 4AK\right). \end{aligned}$$

where θ in $A\left(\theta\right), B\left(\theta\right), C\left(\theta\right)$ and $K\left(\theta\right)$ is eliminated for notational simplicity. Let us denote the quadratic form in the last line by $g\left(\beta_j\right)$. For $\beta_j = 0$, it takes a negative value,

$$g\left(0\right) = B^2 - 4AK = \boldsymbol{D}_G < 0,$$

and the coefficient of β_j^2 is clearly negative for $\theta \in (0, \sqrt{2})$,

$$C^2 - 4A^2 = -12\left(2-\theta^2\right)^3\left(6-\theta^2\right)^3\left(2+\theta^2\right)\left(2+3\theta^2\right) < 0.$$

Hence, $g\left(\beta_j\right)$ is concave in β_j with $g\left(0\right) < 0$. We then have the following:

Lemma 2 $g\left(\beta_j\right) < 0$ *for all* $\beta_j \geq 0$.

Proof. The discriminant of $g\left(\beta_j\right)$ is negative as

$$\begin{aligned}\boldsymbol{D}_g/4 &= \left[B\left(C-2A\right)\right]^2 - \left(C^2-4A\right)\left(B^2-4AD\right), \\ &= -128A\left(2-\theta^2\right)^6\left(6-\theta^2\right)^6\left(2+3\theta^2\right) < 0.\end{aligned}$$

where the direction of the inequality is due to that all factors are positive for $\theta \in (0, \sqrt{2})$. This completes the proof. ■

Lemma 1 indicates that $F_I\left(\beta_i, \beta_j\right)$ is convex in β_i and $F_I\left(0, \beta_j\right) = G_I(\beta_j) < 0$. Lemma 2 implies that $\boldsymbol{D}_F = g\left(\beta_j\right) < 0$ leading to $F_I\left(\beta_i, \beta_j\right) > 0$ for any $\beta_i \geq 0$ and $\beta_j \geq 0$. Therefore, these lemmas imply the following:

Theorem 6 *In a heterogenous duopoly market with no production differentiation, the ambient charge is effective in controlling the total of NPS pollution when the duopoly firms are heterogenous,*

$$\frac{d\boldsymbol{E}_I^C(\theta)}{d\theta} < 0 \text{ for } \theta \in (0, \sqrt{2}).$$

4.2 Duopoly Firms with Product Differentiation

In this section, we move one step forward by replacing $\gamma = 1$ with $\gamma < 1$ and see how product differentiation affects the firms' behavior in case He_2 under Assumption 3 (i.e., $n = 2$ and $\gamma < 1$).[4]

From (7) and (9), the optimal decisions are

$$q_i^{II}(\theta) = \frac{2\left[(4-\theta^2)\beta_i - 2\gamma\beta_j\right] - 2\theta(4-\theta^2-2\gamma)}{(4-\theta^2+2\gamma)\left(4-\theta^2-2\gamma\right)}, \tag{34}$$

$$\phi_i^{II}(\theta) = \frac{\theta\left[2\gamma\beta_j - (4-\theta^2)\beta_i\right] + 2(2+\gamma)(4-\theta^2-2\gamma)}{(4-\theta^2+2\gamma)\left(4-\theta^2-2\gamma\right)}. \tag{35}$$

Equations (10) and (11) with Assumption 3 determine the boundaries of the feasible region of each firm. For firm i, the locus of $\phi_i^{II}(\theta) = 0$ is described by

$$\beta_j = f_0(\beta_i) \equiv \frac{4-\theta^2}{2}\beta_i - \frac{(2+\gamma)(4-\theta^2-2\gamma)}{\gamma\theta}$$

and the locus of $\phi_i^{II}(\theta) = 1$ is by

$$\beta_j = f_1(\beta_i) \equiv \frac{4-\theta^2}{2}\beta_i - \frac{\theta(4-\theta^2-2\gamma)}{2}.$$

[4] Matsumoto and Szidrovszky (2024) have already considered this case. We summarize their procedure and results here.

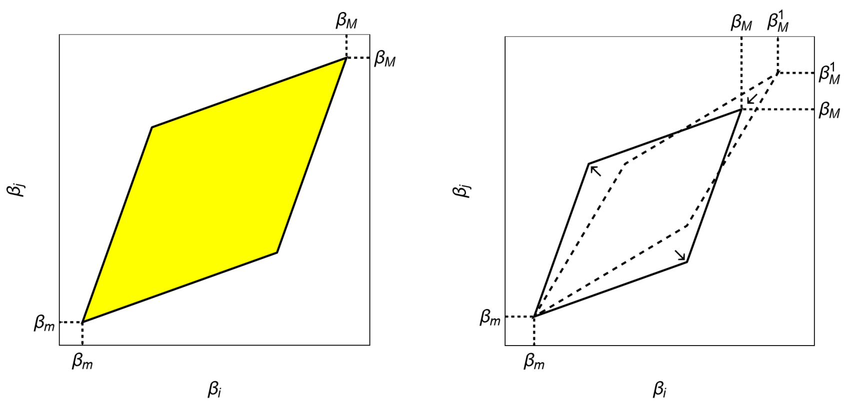
In the same way, we obtain the boundaries for firm j, $\beta_i = f_0(\beta_j)$ and $\beta_i = f_1(\beta_j)$, both of which are solved for β_j to have the alternative forms of firm j's boundaries in terms of β_i,

$$\beta_j = g_0(\beta_i) \equiv \frac{2}{4-\theta^2}\beta_i + \frac{2(2+\gamma)(4-\theta^2-2\gamma)}{\theta\left(4-\theta^2\right)}$$

and

$$\beta_j = g_1(\beta_i) \equiv \frac{2}{4-\theta^2}\beta_i + \frac{\theta(4-\theta^2-2\gamma)}{4-\theta^2}.$$

As in Figure 1, Figure 5(A) constructs the diamond-shaped yellow feasible region in which the upper and lower sides are described by $\beta_j = g_k(\beta_i)$ for $k = 0, 1$ and the left and right sides by $\beta_j = f_k(\beta_i)$ for $k = 0, 1$. In Figure 5(B), the dotted diamond-shaped region with $\gamma = 1$ overlays on the solid diamond-shaped region with $\gamma = 3/5$. Comparing these two regions, we find that decreasing γ transforms the longer and narrower diamond shape to the shorter and wider one. It seems that the solid diamond is larger than the dotted area. However, it is not clear which is larger or whether increasing homogeneity of the good produces a larger feasible region.



(A) $\gamma = 3/5$ (B) $\gamma = 1$ and $\gamma = 3/5$

Figure 5. Feasible regions with $\theta = 4/5$ and various γ

We now consider the ambient charge effect when the firms are heterogenous, $\beta_i \neq \beta_j$. The total amount of NPS pollution is the sum of individual pollutions,

$$\boldsymbol{E}_{II}^{C}(\theta) = E_i^{II}(\theta) + E_j^{II}(\theta) \text{ where } E_k^{II}(\theta) = \phi_k^{II}(\theta)q_k^{II}(\theta) \text{ for } k = i, j.$$

Differentiating it with respect to θ yields

$$\frac{d\boldsymbol{E}_{II}^{C}(\theta)}{d\theta} = -\frac{2F_{II}(\beta i, \beta_j)}{\left(4+2\gamma-\theta^2\right)^3\left(4-2\gamma-\theta^2\right)^3} \tag{36}$$

where the denominator is positive, the numerator $F_{II}(\beta i, \beta_j)$ has the form,

$$F_{II}(\beta i, \beta_j) = A(\gamma, \theta)\left(\beta_i^2 + \beta_j^2\right) + B(\gamma, \theta)\left(\beta_i + \beta_j\right) + C(\gamma, \theta)\beta_i\beta_j + K(\gamma, \theta) \tag{37}$$

with

$$A(\gamma, \theta) = 16\left(4 - \gamma^2\right)\left(4 + \gamma^2\right) + 48\gamma^2\theta^2\left(4 - \theta^2\right) - \theta^4\left(96 - 32\theta^2 + 3\theta^4\right),$$

$$B(\gamma, \theta) = -2\theta\left(4 - 2\gamma - \theta^2\right)^3\left(12 + 6\gamma + \theta^2\right),$$

$$C(\gamma, \theta) = -8\gamma\left\{4\gamma^2\left(3\theta^2 - 4\right) + (4 - \theta^2)^2(4 + 5\theta^2)\right\},$$

$$K(\gamma, \theta) = 4(2 + \gamma)\left(4 + 2\gamma + 3\theta^2\right)\left(4 - 2\gamma - \theta^2\right)^3.$$

Since the domains of the variables are

$$0 < \gamma < 1 \text{ and } 0 < \theta < \sqrt{2},$$

$B(\gamma, \theta) > 0$ and $K(\gamma, \theta) < 0$ are analytically verified, and it is possible to numerically confirm that $A(\gamma, \theta) < 0$ and $C(\gamma, \theta) > 0$, although they have rather complicated forms.

$$F_{II}\left(\beta_i, \beta_j\right) = A\beta_i^2 + \left(B + C\beta_j\right)\beta_i + G_{II}(\beta_j) \tag{38}$$

where

$$G_{II}(\beta_j) = A\beta_j^2 + B\beta_j + K.$$

Notice that the variables γ and θ are omitted only for notational simplicity. Comparing (31) and (32) with (37) and (38), we find that the forms of $F_I\left(\beta_i, \beta_j\right)$ and $F_{II}\left(\beta_i, \beta_j\right)$ are qualitatively the same. We also see that $G_I(\beta_i, \beta_j)$ and $G_{II}(\beta_i, \beta_j)$ are also quadratic polynomials in β_j. Taking the two-step procedure similar to the one used in Section 4.1, we will demonstrate that $F_{II}\left(\beta_i, \beta_j\right)$ is negative for any $\beta_i \geq 0$ and $\beta_j \geq 0$ in three steps (i.e., Lemma 3; Lemma 4 and Theorem 7).

Lemma 3 $G_{II}(\beta_j) < 0$ *for any* $\beta_j \geq 0$.

Proof. $G_{II}(\beta_j)$ is quadratic in β_j and its discriminant is

$$\boldsymbol{D}_G = -\left(4 + 2\gamma - \theta^2\right)\left(4 - 2\gamma - \theta^2\right)h(\gamma, \theta)$$

where

$$h(\gamma, \theta) = 16(2 - \gamma)\left(4 + \gamma^2\right) + \theta^2\left[12\left(4 + 8\gamma - \gamma^2\right) + \theta^2\left(48 - 12\gamma - \theta^2\right)\right].$$

Under Assumptions 1 and 3, $h(\gamma, \theta) > 0$ is confirmed, which then implies $\boldsymbol{D}_G < 0$. With $A < 0$ and $K < 0$, $G_{II}(\beta_j)$ is concave in β_j and has a negative discriminant with $G_{II}(0) = K < 0$. Hence, $G_{II}(\beta_j) < 0$ for all $\beta_j \geq 0$. This completes the proof. ■

Retuning to equation (38), we see that $F_{II}\left(\beta_i,\beta_j\right)$ is concave in β_i as $A<0$ and $F_{II}\left(0,\beta_j\right)=G_{II}(\beta_j)<0$. Its discriminant for β_i is

$$\boldsymbol{D}_F = 4\left(4+2\gamma-\theta^2\right)^3\left(4-2\gamma-\theta^2\right)^3 g(\beta_j) \tag{39}$$

where

$$g(\beta_j) = a_g(\gamma,\theta)\beta_j^2 + 2b_g(\gamma,\theta)\beta_j + c_g(\gamma,\theta) \tag{40}$$

with

$$\begin{aligned} a_g(\gamma,\theta) &= -\left(4+2\gamma+3\theta^2\right)^3\left(4-2\gamma+3\theta^2\right)<0, \\ b_g(\gamma,\theta) &= \theta(4-2\gamma-\theta^2)\left(12+6\gamma+\theta^2\right)>0, \\ c_g(\gamma,\theta) &= -h(\gamma,\theta)<0. \end{aligned}$$

We then have the following:

Lemma 4 $g\left(\beta_j\right)<0$ *for any* $\beta_j\geq 0$.

Proof. $g(\beta_j)$ is quadratic in β_j and its discriminant is

$$\boldsymbol{D}_g = 8\left(4-2\gamma+3\theta^2\right)A<0.$$

Since $g(\beta_j)$ is concave in β_j and $g(0)<0$, we have $g\left(\beta_j\right)<0$ for any $\beta_j\geq 0$. This completes the proof. ■

Lemma 4 implies that $\boldsymbol{D}_F<0$ for any $\beta_j\geq 0$. Hence, from Lemmas 3 and 4, $F_{II}\left(\beta_i,\beta_j\right)$ is negative for any $\beta_i\geq 0$ and $\beta_j\geq 0$. Therefore, we arrive at the following result:

Theorem 7 *In a heterogenous duopoly market with product differentiation, the ambient charge is effective in controlling the total amount of NPS pollution,*

$$\frac{d\boldsymbol{E}_{II}^C(\theta)}{d\theta}<0 \text{ for } \theta\in[0,\sqrt{2}].$$

4.3 Semi-Identical Firms

Although the ambient charge issue becomes too complex in a general setting of $n\geq 3$, we examine the special case of He_4 when the firms are semi-identical heterogeneous in the following sense,

Assumption 7. $\beta_i\neq\beta_j$ and $\beta_j=\beta$ for $j\neq i,\ j=1,2,...i-1,i+1,...,n$.

Quantities (7) and (9) with the specified value of n determine the optimal productions, $q_k^C(\theta)$ and the optimal abatement technology, $\phi_k^C(\theta)$ for $k = i, j$,

$$q_i^C(\theta) = \frac{2\left[(4-\theta^2+2(n-2)\gamma)\beta_i - 2(n-1)\gamma\beta_j\right] - 2\theta(4-\theta^2-2\gamma)}{(4-\theta^2+2(n-1)\gamma)\left(4-\theta^2-2\gamma\right)},$$

$$\phi_i^C(\theta) = \frac{\theta\left[2(n-1)\gamma\beta_j - (4-\theta^2+2(n-2)\gamma)\beta_i\right] + 2\left(2+(n-1)\gamma\right)(4-\theta^2-2\gamma)}{(4-\theta^2+2(n-1)\gamma)\left(4-\theta^2-2\gamma\right)},$$

and

$$q_j^C(\theta) = \frac{2\left[(4-\theta^2)\beta_j - 2\gamma\beta_i\right] - 2\theta(4-\theta^2-2\gamma)}{(4-\theta^2+2(n-1)\gamma)\left(4-\theta^2-2\gamma\right)},$$

$$\phi_j^C(\theta) = \frac{\theta\left[2\gamma\beta_i - (4-\theta^2)\beta_j\right] + 2\left(2+(n-1)\gamma\right)(2-\theta^2-2\gamma)}{(4-\theta^2+2(n-1)\gamma)\left(4-\theta^2-2\gamma\right)}.$$

To determine the feasible region, we solve $\phi_i^C(\theta) = 0$ and $\phi_i^C(\theta) = 1$ for β_j to obtain

$$f_0(\beta_i) = \frac{4+2(n-2)\gamma-\theta^2}{2(n-1)\gamma}\beta_i + \frac{\left(2+(n-1)\gamma\right)(4-\theta^2-2\gamma)}{(n-1)\gamma\theta}$$

and

$$f_1(\beta_i) = \frac{4+2(n-2)\gamma-\theta^2}{2(n-1)\gamma}\beta_i - \frac{\theta(4-\theta^2-2\gamma)}{2(n-1)\gamma}.$$

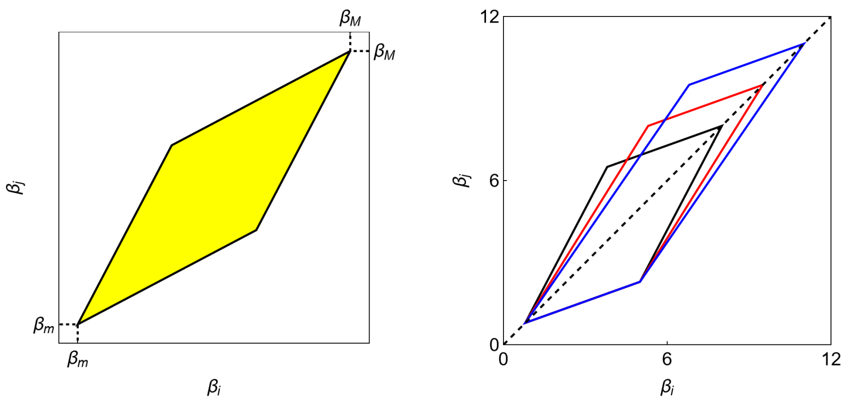
By the same token, we solve $\phi_j^C(\theta) = 0$ and $\phi_j^C(\theta) = 1$ for β_j to obtain

$$g_0(\beta_i) = \frac{2\gamma}{4-\theta^2}\beta_i + \frac{\left(2+(n-1)\gamma\right)(4-\theta^2-2\gamma)}{\theta\left(4-\theta^2\right)}$$

and

$$g_1(\beta_i) = \frac{2\gamma}{4-\theta^2}\beta_i - \frac{\theta(4-\theta^2-2\gamma)}{4-\theta^2}.$$

Figure 6(A) illustrates the feasible region with $\theta = 4/5$, $\gamma = 3/5$ and $n = 3$. Figure 6(B) overlays three feasible regions with $n = 3$ (the black diamond shape), $n = 4$ (the red diamond shape) and $n = 5$ (the blue diamond shape). Each region has the common value of β_m and becomes longer and narrower as n increases.



(A) $n = 3$ (B) $n = 3, 4, 5$

Figure 6. Feasible regions with $\theta = 4/5,\ \gamma = 3/5$ and various values of n

The total amount of NPS pollution is the sum of individual pollutions:

$$\boldsymbol{E}^C(\theta) = E_i^C(\theta) + (n-1)E_j^C(\theta) \text{ where } E_k^C(\theta) = \phi_k^C(\theta)q_k^C(\theta) \text{ for } k = i, j.$$

Differentiating $\boldsymbol{E}^C(\theta)$ with respect to θ yields

$$\frac{d\boldsymbol{E}^C(\theta)}{d\theta} = -\frac{2F(\beta_i, \beta_j, \gamma, \theta, n)}{\left(4 - 2\gamma - \theta^2\right)^3 \left(4 + 2\left(n-1\right)\gamma - \theta^2\right)^3} \tag{41}$$

where

$$F(\beta_i, \beta_j, \gamma, \theta, n) = A_i\beta_i^2 + A_j\beta_j^2 + B_i\beta_i + B_j\beta_j + C_{ij}\beta_i\beta_j + K. \tag{42}$$

The full forms of the coefficients of β_i and β_j and the constant term K in (42) are provided in Appendix as these are too long to present here. Although it is possible to follow the procedure used in the previous sections to show the controllability of the ambient charge, we numerically examine the degree of the controllability in this section. We specify the parameter values as

$$\theta = 4/5 \text{ and } \gamma = 3/5.$$

For $n = 3$, the derivative in (41) is

$$\frac{d\boldsymbol{E}^C(\theta)}{d\theta} = -\frac{125\left(a_i\beta_i^2 + a_j\beta_j^2 + b_i\beta_i + b_j\beta_j + c_{ij}\beta_i\beta_j + k\right)}{7558272}$$

with

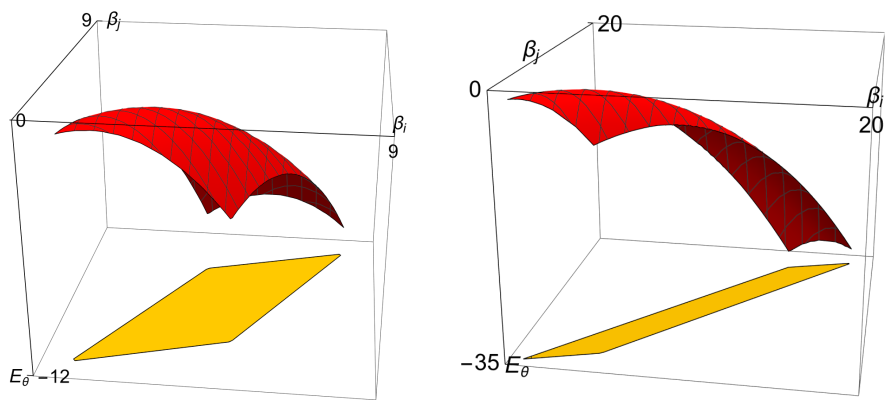
$$a_i = 39515,\ a_j = 44780,\ b_i = -20088,\ b_j = -40176,\ c_{ij} = -98500,\ k = 101088$$

When n increases to 10, the corresponding derivative is

$$\frac{d\boldsymbol{E}^C(\theta)}{d\theta} = -\frac{125\left(a_i\beta_i^2 + a_j\beta_j^2 + b_i\beta_i + b_j\beta_j + c_{ij}\beta_i\beta_j + k\right)}{898327746}$$

with

$$a_i = 39515,\ a_j = 44780,\ b_i = -20088,\ b_j = -40176,\ c_{ij} = -98500,\ k = 101088$$



(A) $n = 3$ (B) $n = 10$

Figure 7. The $d\boldsymbol{E}^C(\theta)/d\theta$ surface over the feasible region

Theorem 8 *Given Assumptions 1 and 2, the ambient charge is effective in controlling the total amount of NPS pollution when the firms are semi-identical heterogenous:*

$$\frac{d\boldsymbol{E}^C}{d\theta} < 0.$$

5 Concluding Remarks

This paper investigates whether the ambient charge can control the total NPS pollution in various circumstances, duopoly or oligopoly with or without product differentiation, when the firms are homogeneous or heterogeneous. To this end, we adopt the game-theoretic approach in which the firms are profit maximizers, taking their competitor's behavior as given. First, the Nash equilibrium is determined, and then a comparative study is performed to validate how a change in the environmental tax rate affects the optimal choices of output, abatement technology, and the total concentration of NPS pollution. It is analytically or numerically demonstrated that the ambient charge effectively controls NPS pollution.

The research reported in this paper can be continued and extended in several directions. Since the regulator's behavior (i.e., environmental tax rate) was given, we should define a social welfare function involving uncertainty and determine the optimal tax rate to maximize this welfare function. Oligopolies with n-firms can be modeled as an $(n + 1)$-player game when the firms and the regulator are the players. Its equilibrium analysis might offer some interesting results. We can also introduce the dynamic extensions of these models with and without time delays. Further, linear price and cost functions are convenient but confine our attention only to a limited area. Replacing them with nonlinear ones will be an appealing extension.

Appendix

Notice first that all calculations in this Appendix are done with Mathematica, version 13. Notice second that all functions defined below depend only on γ, θ and n. Assumptions 1 and 2 restrict the domains of γ and θ to

$$0 < \gamma < 1 \text{ and } 0 < \theta < \sqrt{2}$$

and integer n is greater than or equal to 2. Hence, it is possible to analytically or numerically verify whether these parameter values are positive or negative, even though they have complicated forms.

The full forms of the coefficients of equations (41) and (42) are

$$A_i = -3\theta^8 + a_{i6}\theta^6 + a_{i4}\theta^4 + a_{i2}\theta^2 + a_{i0}$$

with

$$\begin{aligned} a_{i6} &= 32 + 18(n-2)\gamma, \\ a_{i4} &= -\left\{120(n-2)\gamma + 12\left[2(n-2)^2 + n^2\right]\gamma^2 + 96\right\}, \\ a_{i2} &= 24\gamma\left[4(n-2) + 4((n-1)^2+1)\gamma + (n-2)(n^2-(n-1))\gamma^2\right], \\ a_{i0} &= 16(2-\gamma)(2+(n-1)\gamma)\left[4(1+(n-2)\gamma) + (n^2-3(n-1))\gamma^2\right] \end{aligned}$$

and

$$A_j = (n-1)\left[-3\theta^8 + a_{j6}\theta^6 + a_{j4}\theta^4 + a_{j2}\theta^2 + a_{j0}\right]$$

with

$$\begin{aligned} a_{j6} &= 32 + 2(n-2)\gamma, \\ a_{j4} &= 24(n-2)\gamma - 48(n-1)\gamma^2 - 96, \\ a_{j2} &= 24\gamma\left[-4(n-2) + 8(n-1)\gamma + (n-2)(n-1)\gamma^2\right] \\ a_{j0} &= -16(2-\gamma)(2+(n-1)\gamma) + \left[4 + (n-1)\gamma^2\right] \end{aligned}$$

and

$$\begin{aligned} B_i &= -2\theta(4-2\gamma-\theta^2)^3\left[12 + 6(n-1)\gamma + \theta^2\right], \\ B_j &= -2\theta(n-1)(4-2\gamma-\theta^2)^3\left[12 + 6(n-1)\gamma + \theta^2\right], \end{aligned}$$

and

$$C_{ij} = 8(n-1)\gamma(-5\theta^6 + c_4\theta^4 + c_2\theta^2 + c_0),$$

with

$$\begin{aligned} c_4 &= 9\left[4 + (n-2)\gamma\right], \\ c_2 &= -\left[24(n-2)\gamma - 12(n-1)\gamma^2 + 6n^2\gamma^2\right] \\ c_0 &= -4(2-\gamma)(2+(n-1)\gamma)\left(4+(n-2)\gamma\right) \end{aligned}$$

and

$$K = 2n(2+(n-1)\gamma)(4-2\gamma-\theta^2)^3\left[4 + 2(n-1)\gamma + 3\theta^2\right].$$

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