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Abstract

Under linear price and cost functions, the optimal environmental policy is determined for duopolies with product differentiation. In the case of non-point source pollutants, the standard policies cannot be applied since the government has limited information about the individual emission, only the total size of the pollution is known. The firms' decisions are concerned with their outputs and abatement technologies, while the government chooses the uniform pollution tax rate. The optimal decisions are determined in a two-stage process. At the second stage, the firms determine their outputs, taking the technologies and the tax as given. At the first stage, the firms select abatement technologies with given the tax rate, and the government selects optimal tax rate with given choices of the firms. Under asymmetric information, the government constructs the welfare function with uncertainty on the firms' outputs and determines the optimal tax rate by maximizing the welfare expectation and minimizing the welfare variance. Since the best reply of the government has a complicated form, the Nash equilibrium is numerically and graphically solved. It is shown that the optimal ambient charge tax is less than the Pigouvian tax. It is also shown that the ambient charge tax effectively controls the total concentration of NPS pollution.

Keywrods: Emvironmental policy, Ambient charge, NPS pollution, Two-stage game, Cournot duopoly, Multi-objective optimization

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1 Introduction

The optimal choice for the environmental policy is well known for two polar cases of market structure. In a competitive market on one hand, the optimal pollution tax should be set equal to the marginal damage that pollution causes (Pigou (1932)). In a monopoly market on the other hand, it is imposed to be less (Buchanan (1969) and Barnett (1980)). In an oligopoly market, the optimal choice depends on another market's characteristics and the optimal tax could be higher, lower than or equal to the marginal value of the environmental damage. Simpson (1995) shows that the optimal pollution tax rate may exceed the marginal damage if firms have different production costs in a Cournot-duopoly market without product differentiation. Okuguchi (2003) reconsiders Simpson's problem in a general oligopoly market and shows that the optimal tax must be less than the marginal damage in a special case of identical cost functions and the magnitude correlation is indeterminate in the general case. Katsoulacos and Xepapadeas (1995) conclude that the optimal pollution tax falls short of the marginal social value in an exogenous market structure and could exceed it in an endogenous market structure in a homogeneous oligopoly. Poyago-Theotoky (2003) examines the optimal environmental policy in a differentiated goods duopoly and shows that the emission tax is always lower than the marginal damage. Fujiwara (2009) considers how product differentiation affects the optimal pollution tax policies in a free-entry Cournot oligopoly. Lian et al. (2018) re-investigate the optimal policies in a free-entry oligopoly and find that the optimal tax is always less than the marginal environmental damage. Although choosing the optimal environmental tax depends on various factors, one common feature of those studies is that pollution is point source (PS). Since PS pollution has a single discrete source, the government can identify the amount of pollution contaminations and where it comes from. Regulations can be placed upon each polluter.

In this paper, we consider the optimal environmental policy for non-point source (NPS) pollution. It is difficult to identify the sources, the sizes and the origins of any specific NPS pollution. The government can observe only the ambient concentration of pollutants associated with unobserved individual emissions. Due to informational asymmetries, it is also difficult to apply traditional environmental policy instruments such as emission taxes or tradable permits for controlling NPS pollution. Segerson (1988) advocates an ambient scheme in the form of an ambient tax. Accordingly, the government sets a standard environmental level of pollution and imposes a uniform tax on all possible polluters if the total concentration is above the standard and provides uniform subsidies if it is below the standard. Our main question concerns two issues: selecting the optimal ambient tax rate and confirming whether the ambient tax can control pollution. To proceed, we assume a Cournot duopoly in which goods are differentiated and production costs are different. Then we construct a two-stage game: in the first stage, the government sets the optimal tax rate of ambient charge, and the firms select their abatement technologies simultaneously and in the second stage, the firms determine their outputs.

The paper is organized as follows. Section 2 builds the basic model. Section 3 provides the results of the government and the firms at the first stage. Section 4 determines Nash equilibrium and presents a comparative analysis. Section 5 considers the ambient tax effects on the ambient concentration. Section 6 presents concluding remarks and outlines further research directions.

2 Basic Model

We consider the following two-stage game between the profit maximizing duopoly firms and the welfare maximizing government. At the first-stage, the government sets the ambient charge rate for emission$s$ of NPS pollution. The firms decide the levels of the abatement technology. At the second-stage, the firms select their outputs, taking the ambient charge tax and the abatement technologies. As the usual procedure, we solve this two-stage game backwardly. In this section, the model construction and the analysis of the second-stage will be done.

We first construct a simple Cournot duopoly model. The two firms indexed by i and j produce differentiated goods and face linear price (inverse demand) functions. For firm i, the price function is

$$p_i(q_i, q_j) = \alpha_i - q_i - \gamma q_j \tag{1}$$

and for firm j, it is

$$p_j(q_i, q_j) = \alpha_j - \gamma q_i - q_j \tag{2}$$

where q_k and $\alpha_k > 0$ are the output and the maximum price of firm k for $k = i, j$. The parameter γ measures the degree of product differentiation and is subject to $0 \leq \gamma \leq 1$ in which two goods are substitutes if $0 < \gamma < 1$, homogenous if $\gamma = 1$ and independent if $\gamma = 0$.

Pollution is a by-product of the production process. It can be assumed that one unit of production emits one unit of pollution (after some appropriate choice of units). Let ϕ_k denote the pollution abatement technology of firm k where $0 \leq \phi_k \leq 1$ with the pollution-free technology if $\phi_k = 0$ and a fully-discharge technology if $\phi_k = 1$. With ϕ_k, firm k eliminates $(1 - \phi_k)q_k$ -pollution and thus discharges $\phi_k q_k$. The government determines an environmental standard level of pollution, $\bar{E}$, and imposes a uniform tax on the polluters if the concentration is above the standard and pays uniform subsidies if below. Both firms have linear production costs with c_i and c_j denoting their marginal production costs. We allow for cost asymmetry ($c_i \neq c_j$) and demand asymmetry ($\alpha_i \neq \alpha_j$).

Firm k assumes that the competitor will unchange its choice of quantity and chooses q_k to maximize its profit defined by

$$\pi_k(q_k, \phi_k) = (\alpha_k - q_k - \gamma q_{\bar{k}})\, q_k - c_k q_k - \eta(1 - \phi_k)^2 - \theta(\phi_k q_k + \phi_{\bar{k}} q_{\bar{k}} - \bar{E}). \tag{3}$$

Here $\bar{k}$ means the competitor of firm k (e.g., if $k = i$, then $\bar{k} = j$), c_k is the marginal production cost and θ is the ambient charge tax rate that will be often called shortly the ambient tax or the ambient charge hereafter. $\eta(1 - \phi_i)^2$

denotes the installation cost of technology. Notice that $\eta = 0$ if the ambient technology has already been installed and $\eta = 1$ if the ambient technology is a choice variable. The rate θ is measured in a monetary unit per emission (e.g., dollar/ton, yen/kg, etc.). It is positive and can be larger than unity. To guarantee a positive profit in the case of no pollutions, we assume $\alpha_k > c_k$. We state some of these assumptions more formally.

Assumption 1. (1) $0 < \gamma < 1$; (2) $\alpha_k > c_k$ for $k = i, j$.

We start with the analysis of firm i. At the second-stage, firm i determines a production level of output to maximize the profit, taking the ambient tax and its abatement technology as given (i.e., $\eta = 0$ in (3)). Differentiating (3) with respect to q_i presents the first-order condition for an interior solution,

$$\frac{\partial \pi_i}{\partial q_i} = \alpha_i - 2q_i - \gamma q_j - c_i - \theta\phi_i = 0. \tag{4}$$

The first-order condition for firm j is similarly derived. Concavity of the profit functions ensures satisfaction of the second-order conditions for both firms. The optimal levels of output can be obtained by solving simultaneously these two first-order conditions, which can be rewritten as

$$\begin{aligned} 2q_i + \gamma q_j &= \alpha_i - c_i - \theta\phi_i, \\ \gamma q_i + 2q_j &= \alpha_j - c_j - \theta\phi_j. \end{aligned} \tag{5}$$

The difference, $\alpha_k - c_k > 0$, is positive due to Assumption 1(2) and it is a measure of the market size for firm k. Denoting the difference as β_k and solving (5) yield the following Nash equilibrium quantities at the second-stage:

$$\begin{aligned} q_i^*(\theta, \phi_i, \phi_j) &= \frac{2\beta_i - \gamma\beta_j - \theta\left(2\phi_i - \gamma\phi_j\right)}{4 - \gamma^2}, \\ q_j^*(\theta, \phi_i, \phi_j) &= \frac{2\beta_j - \gamma\beta_i - \theta\left(2\phi_j - \gamma\phi_i\right)}{4 - \gamma^2}. \end{aligned} \tag{6}$$

Since the denominator of each equation in (6) is positive, the optimal outputs are non-negative if the numerators are non-negative,

$$\phi_j \geq b_L(\phi_i) \text{ for } q_i^* \geq 0 \text{ and } \phi_j \leq b_U(\phi_i) \text{ for } q_j^* \geq 0$$

where

$$b_L(\phi_i) = \frac{2}{\gamma}\phi_i - \frac{2\beta_i - \gamma\beta_j}{\gamma\theta} \text{ and } b_U(\phi_i) = \frac{\gamma}{2}\phi_i + \frac{2\beta_j - \gamma\beta_i}{2\theta}. \tag{7}$$

It is clear that $\gamma/2 < 1 < 2/\gamma$ due to Assumption 1(1). We let ϕ_i^0 and ϕ_j^0 be the x-intercept of $b_L(\phi_i)$ and the y-intercept of $b_U(\phi_i)$,

$$\phi_i^0 = \frac{2\beta_i - \gamma\beta_j}{2\theta} \text{ and } \phi_j^0 = \frac{2\beta_j - \gamma\beta_i}{2\theta}.$$

If these intercepts are non-negative, then there is a set of (β_i, β_j) for which the outputs in (6) are non-negative. We will confine our analysis to the case in which β_i, β_j and γ are subject to the following inequalities:

Assumption 2. $\frac{\gamma}{2}\beta_i \leq \beta_j \leq \frac{2}{\gamma}\beta_i$.

Hence, summarizing the results allow us to state the followings:

Lemma 1 *Given Assumptions 1 and 2, the optimal outputs are non-negative,*

$$q_i^*(\theta, \phi_i, \phi_j) \geq 0 \text{ and } q_j^*(\theta, \phi_i, \phi_j) \geq 0$$

if the abatement technologies satisfy the following relations,

$$b_L(\phi_i) \leq \phi_j \leq b_U(\phi_i).$$

At the second-stage, where the technology levels are fixed, differentiating the optimal outputs with respect to θ reveals that individual output responses depend on the technology levels and the product-differentiation degree,

$$\frac{\partial q_i^*}{\partial \theta} = \frac{\gamma\phi_j - 2\phi_i}{4-\gamma^2} \text{ and } \frac{\partial q_j^*}{\partial \theta} = \frac{\gamma\phi_i - 2\phi_j}{4-\gamma^2}.$$

Roughly speaking, the individual responses are non-positive if the firms have similar technologies,

$$\frac{\partial q_i^*}{\partial \theta} \leq 0 \text{ and } \frac{\partial q_j^*}{\partial \theta} \leq 0 \text{ if } \frac{\gamma}{2}\phi_i \leq \phi_j \leq \frac{2}{\gamma}\phi_i \tag{8}$$

and the firm with a much more efficient technology positively responses to a tax change, which is often called a *perverse effect* of θ,

$$\begin{aligned} &\frac{\partial q_i^*}{\partial \theta} > 0 \text{ and } \frac{\partial q_j^*}{\partial \theta} < 0 \text{ if } \phi_j > \frac{2}{\gamma}\phi_i \\ &\frac{\partial q_i^*}{\partial \theta} < 0 \text{ and } \frac{\partial q_j^*}{\partial \theta} > 0 \text{ if } \phi_j < \frac{\gamma}{2}\phi_i. \end{aligned} \tag{9}$$

Although the effect on the individual responses are indeterminate in general, (8) and (9) indicate no possibility that both firms have the perverse effect simultaneously. Notice that the government is unable to observe these individual responses, however, able to observe the total responses. The impact on the total output is always negative,

$$\frac{\partial q_i^*}{\partial \theta} + \frac{\partial q_j^*}{\partial \theta} = -\frac{\phi_i + \phi_j}{2+\gamma} < 0.$$

This inequality means that the negative individual response dominates the positive individual response, irrespective of the technology differences.

Theorem 1 *At the second-stage in which the abatement technologies are given, the ambient tax is effective in controlling the total amount,* $E^*(\theta)$*, of NPS pollution,*

$$\frac{dE^*(\theta)}{d\theta} < 0$$

where $E^*(\theta)$ *denotes the sum of individual emissions,*

$$E^*(\theta) = \phi_i q_i^*(\theta, \phi_i, \phi_j) + \phi_j q_j^*(\theta, \phi_i, \phi_j).$$

Proof. Differentiating $E^*(\theta)$ with respect to θ yields

$$\frac{dE^*(\theta)}{d\theta} = -\frac{2}{4-\gamma^2}\left[\left(\phi_i - \phi_j\right)^2 + (2-\gamma)\,\phi_i\phi_j\right] < 0$$

where the inequality is due to Assumption 1(1). ■

3 Optimal Abatement Technology

At the first-stage, the firms and the government make their decisions simultaneously. This section concerns firm's optimal choice of the abatement technology. The firms determine their abatement technologies, ϕ_i and ϕ_j, to maximize their profits, taking θ as given. To define the profit at the first-stage, we substitute the optimal outputs in (6) into the profit function (3) to obtain the reduced form of firm i's profit,

$$\pi_i^*(\phi_i, \phi_j) = \left(\alpha_i - q_i^* - \gamma q_j^*\right) q_i^* - c_i q_i^* - (1-\phi_i)^2 - \theta\left[\sum_{k=1}^{2} \phi_k q_k^* - \bar{E}\right] \quad (10)$$

where the variables, θ, ϕ_i, ϕ_j, of the optimal outputs are omitted for notational simplicity. Differentiating (10) with respect to ϕ_i yields the first-order condition for firm i's interior optimal solution of the abatement technology,

$$\frac{\partial \pi_i^*}{\partial \phi_i} = \frac{\partial \pi_i^*}{\partial q_i}\frac{\partial q_i^*}{\partial \phi_i} + \frac{\partial \pi_i^*}{\partial q_j}\frac{\partial q_j^*}{\partial \phi_i} + \left.\frac{\partial \pi_i^*}{\partial \phi_i}\right|_{q_i^*, q_j^*: const} = 0 \quad (11)$$

where

$$\frac{\partial \pi_i^*}{\partial q_i} = \alpha_i - 2q_i^* - \gamma q_j^* - c_i - \theta\phi_i = 0,$$

$$\frac{\partial \pi_i^*}{\partial q_j} = -\gamma q_i^* - \theta\phi_j,$$

$$\frac{\partial q_i^*}{\partial \phi_i} = -\frac{2\theta}{4-\gamma^2},$$

$$\frac{\partial q_j^*}{\partial \phi_i} = \frac{\gamma\theta}{4-\gamma^2},$$

$$\left.\frac{\partial \pi_i^*}{\partial \phi_i}\right|_{q_i^*, q_j^*: const} = 2(1-\phi_i) - \theta q_i^*.$$

Since the further differentiation of the profit function yields

$$\frac{\partial^2 \pi_i^*}{\partial \phi_i^2} = -2\left[\left(4-\gamma^2\right)^2 - (2\theta)^2\right],$$

the second-order condition (SOC henceforth) holds if

$$\theta < \tilde{\theta}(\gamma) = \frac{4-\gamma^2}{2}. \tag{12}$$

Substituting q_i^* and q_j^* in (6) into (10) and rearranging the terms simplify the form of the first-order condition for firm i,

$$2\left[\left(4-\gamma^2\right)^2 - (2\theta)^2\right]\phi_i + \left(8-\gamma^2\right)\gamma\theta^2\phi_j = 4\theta\left(\gamma\beta_j - 2\beta_i\right) + 2\left(4-\gamma^2\right)^2.$$

In the same way, the first-order condition for firm j is obtained as

$$\left(8-\gamma^2\right)\gamma\theta^2\phi_i + 2\left[\left(4-\gamma^2\right)^2 - (2\theta)^2\right]\phi_j = 4\theta\left(\gamma\beta_i - 2\beta_j\right) + 2\left(4-\gamma^2\right)^2.$$

These are put in a matrix form,

$$\begin{pmatrix} 2\left[(4-\gamma^2)^2 - (2\theta)^2\right] & \left(8-\gamma^2\right)\gamma\theta^2 \\ \left(8-\gamma^2\right)\gamma\theta^2 & 2\left[(4-\gamma^2)^2 - (2\theta)^2\right] \end{pmatrix} \begin{pmatrix} \phi_i \\ \phi_j \end{pmatrix} = \begin{pmatrix} a_i \\ a_j \end{pmatrix} \tag{13}$$

where

$$a_i = 4\theta\left(\gamma\beta_j - 2\beta_i\right) + 2\left(4-\gamma^2\right)^2 \text{ and } a_j = 4\theta\left(\gamma\beta_i - 2\beta_j\right) + 2\left(4-\gamma^2\right)^2.$$

Solving equation (13) yields the optimal abatement technologies of firms i and j,

$$\begin{aligned}\phi_i^e(\theta) &= \frac{2\left\{2\theta\left[(\gamma A+2B)\beta_j-(2A+\gamma B)\beta_i\right]+\left(4-\gamma^2\right)^2(A-B)\right\}}{A^2-B^2}\\ \phi_j^e(\theta) &= \frac{2\left\{2\theta\left[(\gamma A+2B)\beta_i-(2A+\gamma B)\beta_j\right]+\left(4-\gamma^2\right)^2(A-B)\right\}}{A^2-B^2}\end{aligned} \tag{14}$$

where

$$A=2\left[(4-\gamma^2)^2-(2\theta)^2\right] \text{ and } B=\left(8-\gamma^2\right)\gamma\theta^2.$$

$A>0$ and $B>0$ by (12), Assumption 1(1) and Theorem 1. The difference between A and B is

$$A-B=(2+\gamma)\left(4+2\gamma-\gamma^2\right)\left[\bar{\theta}(\gamma)^2-\theta^2\right]$$

with

$$\bar{\theta}(\gamma)=\frac{(2-\gamma)\sqrt{4+2\gamma}}{\sqrt{4+2\gamma-\gamma^2}}. \tag{15}$$

It is apparent that $A-B>0$ if $\theta<\bar{\theta}(\gamma)$. Although this inequality condition is stronger than the SOC, we assume it to make the following analysis simpler:

Assumption 3. $\theta<\bar{\theta}(\gamma)$ for $\gamma\in(0,1)$.

We check the conditions for $0\le\phi_i^e(\theta)\le1$. In the first equation of (14), the denominator is positive by Assumption 3 and the numerator is non-negative if the blacketed term is non-negative. Solving it for β_j, we find that $\phi_i^e(\theta)\ge0$ if

$$\beta_j\ge f_1(\beta_i) \tag{16}$$

where

$$f_1(\beta_i)=\frac{2A+\gamma B}{\gamma A+2B}\beta_i-\frac{(4-\gamma^2)^2(A-B)}{2\theta(\gamma A+2B)}.$$

The inequality of $\phi_i^e\le1$ holds if the denominator is greater than or equal to the numerator or

$$\beta_j\le f_2(\beta_i). \tag{17}$$

where

$$f_2(\beta_i)=\frac{2A+\gamma B}{\gamma A+2B}\beta_i+\frac{\left[A+B-2(4-\gamma^2)^2\right](A-B)}{4\theta(\gamma A+2B)}.$$

Hence ϕ_i^e is positive and less than or equal to unity if (16) and (17) hold. In the same way, the conditions for $0\le\phi_j^e\le1$ can be written as

$$f_1\left(\beta_j\right)\le\beta_i\le f_2\left(\beta_j\right)$$

or

$$f_2^{-1}(\beta_i) \leq \beta_j \leq f_1^{-1}(\beta_i) \tag{18}$$

where

$$f_1^{-1}(\beta_i) = \frac{\gamma A + 2B}{2A + \gamma B}\beta_i + \frac{(4-\gamma^2)^2(A-B)}{2\theta(2A+\gamma B)}$$

and

$$f_2^{-1}(\beta_i) = \frac{\gamma A + 2B}{2A + \gamma B}\beta_i - \frac{\left[A + B - 2(4-\gamma^2)^2\right](A-B)}{4\theta(2A+\gamma B)}$$

The results concerning the optimal technologies are summarized as follows:

Lemma 2 *Given Assumptions 1, 2 and 3, the optimal abatement technologies are non-negative and not greater than unity,*

$$0 \leq \phi_i^e(\theta) \leq 1 \textit{ and } 0 \leq \phi_j^e(\theta) \leq 1,$$

if the following conditions for β_i and β_j are satisfied,

$$f_1(\beta_i) \leq \beta_j \leq f_2(\beta_i) \textit{ and } f_2^{-1}(\beta_i) \leq \beta_j \leq f_1^{-1}(\beta_i).$$

In Figure 1 with $\gamma = 0.6$ and $\theta = 0.8$, Assumption 2 is satisfied in the gray region and so are the following inequality conditions in the union of the yellow and green regions[1],

$$0 \leq \phi_i^e(\theta) \leq 1 \text{ and } 0 \leq \phi_j^e(\theta) \leq 1.$$

The lower and upper dashed-blue lines are described by $\beta_j = f_2^{-1}(\beta_i)$ and $\beta_j = f_1^{-1}(\beta_i)$ (i.e., $\phi_j^e(\theta) = 1$ and $\phi_j^e(\theta) = 0$) whereas the left and right dashed-red lines by $\beta_j = f_2(\beta_i)$ and $\beta_j = f_1(\beta_i)$ (i.e., $\phi_i^e(\theta) = 1$ and $\phi_i^e(\theta) = 0$). Solving $f_2(\beta_i) = f_2^{-1}(\beta_i)$ and $f_1(\beta_i) = f_1^{-1}(\beta_i)$ presents the minimum value β_m and the maximum value β_M,

$$\beta_m = \frac{\left(4 - 2\gamma - \gamma^2\right)\theta}{4} \text{ and } \beta_M = \frac{(2-\gamma)(2+\gamma)^2}{2\theta}.$$

Substituting $\phi_i^e(\theta)$ and $\phi_j^e(\theta)$ of (14) into q_i^* and q_j^* in (6) determines the optimal production with the optimal abatement technology,

$$q_k^e(\theta) = q_k^*(\theta, \phi_i^e(\theta), \phi_j^e(\theta)) \text{ for } k = i, j$$

that can be written as

$$q_k^e(\theta) = \frac{M\beta_k - N\beta_{\bar{k}} - L}{(4-\gamma^2)(A^2 - B^2)} \text{ for } k = i, j \tag{19}$$

where

$$L = 2(A-B)(2-\gamma)^3(2+\gamma)^2\theta,$$

[1] The meanings of the color regions are explained shortly below.

$$M = 2(A^2 - B^2) + 4(4A + 4B\gamma + A\gamma^2)\theta^2,$$
$$N = \gamma(A^2 - B^2) + 4(4B + 4A\gamma + B\gamma^2)\theta^2.$$

The lines of $q_i^e(\theta) = 0$ and $q_j^e(\theta) = 0$ are described by the common boundaries of the yellow and green regions in Figure 1.[2] The non-negativity of $q_k^e(\theta)$ is ensured in the yellow region whereas $q_k^e(\theta)$ is negative in the green region. Notice that $0 \leq \phi_k^e(\theta) \leq 1$ holds in both regions.

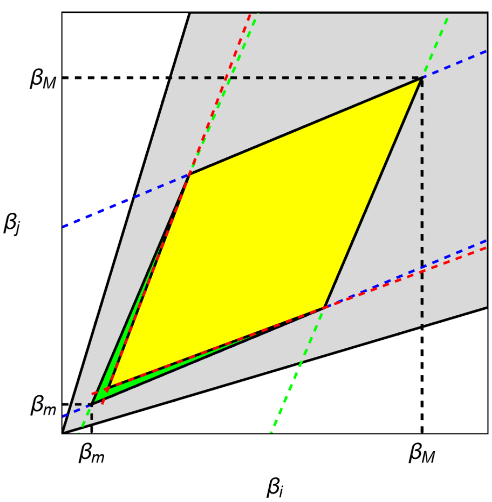


Figure1. Feasible regions of $0 \leq \phi_k^e(\theta) \leq 1$ and $q_k^e(\theta) \geq 0$ for $k = i, j$ under $\gamma = 0.6$ and $\theta = 0.8$

Some properties of the optimal decisions are examined. From (14), a difference of $\phi_i^e(\theta)$ from $\phi_j^e(\theta)$ depends on the market-size difference,

$$\phi_i^e(\theta) - \phi_j^e(\theta) = -\frac{4\theta(2+\gamma)}{A-B}\left(\beta_i - \beta_j\right).$$

An output difference can be written as

$$q_i^e(\theta) - q_j^e(\theta) = \frac{(A-B) + 4\theta^2(2+\gamma)}{(2-\gamma)(A-B)}\left(\beta_i - \beta_j\right)$$

where the first factor on the right-hand side is positive. Those results are summarized:

[2] The zero production lines are described by

$$\beta_j = \frac{M}{N}\beta_i - \frac{L}{N} \ \text{(i.e., } q_i^e(\theta) = 0\text{)}$$

and

$$\beta_j = \frac{N}{M}\beta_i + \frac{L}{M} \ \left(\text{i.e., } q_j^e(\theta) = 0\right).$$

Since $M > N > 0$, the $q_i^e(\theta) = 0$ line is steeper than the $q_j^e(\theta) = 0$ line.

Theorem 2 *Given Assumptions 1, 2 and 3, the firm with larger market size adopts more efficient abatement technology and produces more output, and if the market sizes are equal, then the two firms' choices are identical,*

$$\phi_i^e(t) \lesseqgtr \phi_j^e(\theta) \text{ and } q_i^e(\theta) \gtreqless q_j^e(\theta) \text{ according to } \beta_i \gtreqless \beta_j.$$

We now turn attention to the effects caused by a change in the ambient tax on the optimal abatement technology. Differentiating $\phi_i^e(\theta)$ and $\phi_j^e(\theta)$ with respect to θ yields

$$\begin{aligned} \frac{d\phi_i^e(\theta)}{d\theta} &= 4\left(\frac{4-\gamma^2}{A^2-B^2}\right)^2 \left(\ell - m\beta_i + n\beta_j\right) \\ \frac{d\phi_j^e(\theta)}{d\theta} &= 4\left(\frac{4-\gamma^2}{A^2-B^2}\right)^2 \left(\ell - m\beta_j + n\beta_i\right) \end{aligned} \tag{20}$$

where

$$\ell = (2-\gamma)(2+\gamma)^2(4-2\gamma-\gamma^2)\theta\left[2(2-\gamma)^2(2+\gamma)-(4+2\gamma-\gamma^2)\theta^2\right]^2,$$

$$m = \left(4-\gamma^2\right)\left(m_0 - m_2\theta^2 - m_4\theta^4 + m_6\theta^6\right),$$

$$n = 2\gamma\left(n_0 + n_2\theta^2 - n_4\theta^4 - n_6\theta^6\right)$$

and the forms of m_k and n_k for $k = 0, 2, 4, 6$ are given in the Appendix 1. It is confirmed that $\ell > 0$, $m > 0$, $n > 0$ and $m > n$ for $0 < \gamma < 1$ and $0 < \theta < 2$. Hence the signs of the derivatives are

$$\begin{aligned} \frac{d\phi_i^e(\theta)}{d\theta} &\gtreqless 0 \text{ according to } \beta_j \gtreqless \frac{m}{n}\beta_i - \frac{\ell}{n}, \\ \frac{d\phi_j^e(\theta)}{d\theta} &\gtreqless 0 \text{ according to } \beta_j \lesseqgtr \frac{n}{m}\beta_i + \frac{\ell}{m}. \end{aligned} \tag{21}$$

The inequality $m > n$ implies that the $d\phi_i^e(\theta)/d\theta = 0$ line is steeper than the $d\phi_j^e(\theta)/d\theta = 0$ line and both are illustrated as dotted-black lines in Figure 2.

We now examine the ambient effect on the optimal outputs. Differentiating (19) with respect to θ presents the following forms of the derivatives,

$$\begin{aligned} \frac{dq_i^e(\theta)}{d\theta} &= \frac{2(4-\gamma^2)^3}{\left(A^2-B^2\right)^2}\left(-\bar{\ell} + \bar{m}\beta_i - \bar{n}\beta_j\right), \\ \frac{dq_j^e(\theta)}{d\theta} &= \frac{2\left(4-\gamma^2\right)^3}{\left(A^2-B^2\right)^2}\left(-\bar{\ell} + \bar{m}\beta_j - \bar{n}\beta_i\right) \end{aligned} \tag{22}$$

where the constant term, the coefficients of β_i and β_j are

$$\bar{\ell} = \left[2(2-\gamma)(2+\gamma)^2 - (4-2\gamma-\gamma^2)\theta^2\right]\left[2(2-\gamma)^2(2+\gamma) - (4+2\gamma-\gamma^2)\theta^2\right]^2,$$

$$\bar{m} = 8\theta\left[\bar{m}_0 - \bar{m}_2\theta^2 + \bar{m}_4\theta^4\right],$$

$$\bar{n} = 32\gamma(4-\gamma^2)\left(4(4-\gamma^2)\theta + \gamma^2\theta^3 - \theta^5\right)$$

and the forms of $\bar{m}_k$ for $k = 0, 2, 4$ are also given in the Appendix 1. It is confirmed that $\bar{\ell} > 0$, $\bar{m} > 0$, $\bar{n} > 0$ and $\bar{m} > \bar{n}$ for $0 < \gamma < 1$ and $0 < \theta < 2$. Hence the sigh of the derivatives are

$$\begin{aligned} \frac{dq_i^e(\theta)}{d\theta} &\gtreqless 0 \text{ according to } \beta_j \lesseqgtr \bar{g}_i(\beta_i) = \frac{\bar{m}}{\bar{n}}\beta_i - \frac{\bar{\ell}}{\bar{n}}, \\ \frac{dq_j^e(\theta)}{d\theta} &\gtreqless 0 \text{ according to } \beta_j \gtreqless \bar{g}_j(\beta_i) = \frac{\bar{n}}{\bar{m}}\beta_i + \frac{\bar{\ell}}{\bar{m}}. \end{aligned} \tag{23}$$

The inequality $\bar{m} > \bar{n}$ implies that the $dq_i^e(\theta)/d\theta = 0$ line is steeper than the $dq_j^e(\theta)/d\theta = 0$ line and both are illustrated as two dotted-red lines in Figure 2. As a result, the yellow region in Figure 2 is divided into 9 subregions and each subregion is numbered.[3]

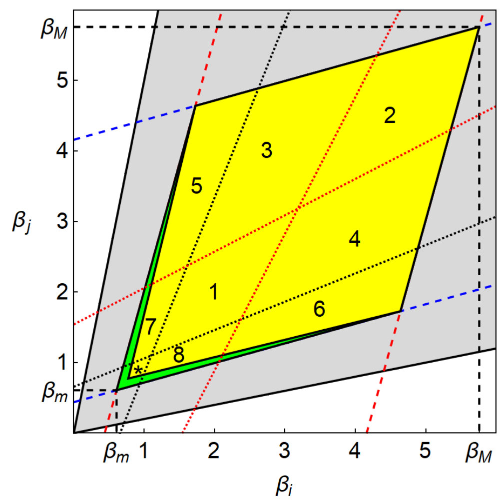


Figure 2. Division of the yellow region by the dotted-black lines of $d\phi_k^e/d\theta = 0$ and the dotted-red lines of $dq_k^e/d\theta = 0$ for $k = i, j$ where $\gamma = 0.4$ and $\theta = 0.8$

4 Ambient Charge Effects on Pollution

In this section we raise a question on whether the ambient charge can control the individual and total amount of NPS pollution. The individual emission is defined as

$$E_k^e(\theta) = \phi_k^e(\theta) q_k^e(\theta) \text{ for } k = i, j \tag{24}$$

and the ambient charge effect is obtained by differentiating $E_k^e(\theta)$ for θ,

$$\frac{dE_k^e(\theta)}{d\theta} = \frac{d\phi_k^e(\theta)}{d\theta} q_k^e(\theta) + \phi_k^e(\theta) \frac{dq_k^e(\theta)}{d\theta}$$

[3] The quadrangle in the lower-left corner of the yellow region is marked "*", meaning "9" where its size is too small to write "9."

where the signs of the derivatives in each subregion are summarized in Table 1 in which R_n means the subregion with number n in Figure 2. We then have the followings:

(1) In R_1, the ambient charge is effective for both firms,

$$\frac{dE_i^e(\theta)}{d\theta} < 0 \text{ and } \frac{dE_j^e(\theta)}{d\theta} < 0.$$

(2) In addition, it is effective for firm i in R_3 and R_8 while it is effective for firm j in R_4 and R_7.

(3) There is no possibility that an increase in the ambient charge increases individual emissions of both firms,

$$\frac{dE_i^e(\theta)}{d\theta} > 0 \text{ and } \frac{dE_j^e(\theta)}{d\theta} > 0.$$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9
$d\phi_i^e/d\theta$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$+$
$dq_i^e/d\theta$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$-$
$dE_i^e/d\theta$	$-$	?	$-$	?	?	?	?	$-$	?
$d\phi_j^e/d\theta$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$+$
$dq_j^e/d\theta$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$-$	$-$
$dE_i^e/d\theta$	$-$	?	?	$-$	?	?	$-$	?	?

Table 1. Summary of the individual responses in the yellow regions

Concerning the ambient charge effect on the total emission, we find one clear result from Table 1 that it is effective in R_1 and seems to be ambiguous in any other regions. To proceed further, we consider the effect from a different view point. Since the total emission is defined as the sum of the individual emissions,

$$\boldsymbol{E}(\theta) = E_i^e(\theta) + E_j^e(\theta), \tag{25}$$

the derivative of $\boldsymbol{E}(\theta)$ with respect to θ describes the ambient charge effect on the total amount that has, after arranging the terms, the following form,

$$\frac{d\boldsymbol{E}(\theta)}{d\theta} = 4\frac{G\left(\beta_i, \beta_j, \gamma, \theta\right)}{\left[F\left(\gamma, \theta\right)\right]^3} \tag{26}$$

where $G\left(\beta_i, \beta_j, \gamma, \theta\right)$ and $F\left(\gamma, \theta\right)$ have the following forms,

$$A\left(\gamma, \theta\right)\left(\beta_i^2 + \beta_j^2\right) + B\left(\gamma, \theta\right)\left(\beta_i + \beta_j\right) + C\left(\gamma, \theta\right)\beta_i\beta_j + D\left(\gamma, \theta\right) \tag{27}$$

and

$$\left[2(2-\gamma)^2(2+\gamma)-\left(4+2\gamma-\gamma^2\right)\theta^2\right]\left[2(2-\gamma)(2+\gamma)^2-\left(4-2\gamma-\gamma^2\right)\theta^2\right] \tag{28}$$

The full forms of $A(\gamma,\theta)$, $B(\gamma,\theta)$, $C(\gamma,\theta)$ and $D(\gamma,\theta)$ are provided in Appendix 2 as they are too long to present here.

The denominator of (26) is positive because $F(\gamma,\theta)$ is positive under Assumption 3 and the following inequality,

$$\theta < \bar{\theta}(\gamma) < \frac{(2+\gamma)\sqrt{4-2\gamma}}{\sqrt{4-2\gamma-\gamma^2}}.$$

Although the numerator, $G\left(\beta_i,\beta_j,\gamma,\theta\right)$, has a long and clumsy form, we will show its negativity by taking a step-by-step approach. We give an outline of our approach. At the first step, we transform $G\left(\beta_i,\beta_j,\gamma,\theta\right)$ to a quadratic form in β_i and denote it as $g_i(\beta_i)$,

$$g_i(\beta_i) = A_i(\gamma,\theta)\beta_i^2 + B_i(\gamma,\theta)\beta_i + C_i(\gamma,\theta) \tag{29}$$

with $A_i(\gamma,\theta) < 0$ and $C_i(\gamma,\theta) < 0$. At the second step, we show that the discriminant of $g_i(\beta_i)$ is factored as

$$\boldsymbol{D}_i(\gamma,\theta) = K(\gamma,\theta)\, g_j(\beta_j) \text{ with } K(\gamma,\theta) > 0 \tag{30}$$

where $g_j(\beta_j)$ is quadratic in β_j,

$$g_j(\beta_j) = A_j(\gamma,\theta)\beta_j^2 + B_j(\gamma,\theta)\beta_i + C_j(\gamma,\theta) \tag{31}$$

with $A_j(\gamma,\theta) < 0$ and $C_j(\gamma,\theta) < 0$. The forms of $A_j(\gamma,\theta)$, $B_j(\gamma,\theta)$ and $C_j(\gamma,\theta)$ are given in Appendix 3. Finally, at the third step, we establish that the discriminant of $g_j(\beta_j)$ is negative. We then trace back each step. In particular, $g_j(\beta_j) < 0$ implies $\boldsymbol{D}_i(\gamma,\theta) < 0$ through (30). Then this negative discriminant with $A_i(\gamma,\theta) < 0$ and $C_i(\gamma,\theta) < 0$ leads to $g_i(\beta_i) < 0$ or $G\left(\beta_i,\beta_j,\gamma,\theta\right) < 0$ through (29). We follow the approach more precisely in Appendix 4. These results lead to the following:

Theorem 3 *Under Assumptions 1, 2 and 3, the ambient charge is effective in controlling the total amount of NPS pollution,*

$$\frac{d\boldsymbol{E}(\theta)}{d\theta} < 0\,.$$

In Figure 3 with $\gamma = 0.6$ and $\theta = 0.8$, the red surface is described by $d\boldsymbol{E}(\theta)/d\theta$ and is located below the upper surface of the cube. The dark yellow quadrangle on the bottom surface is the feasible region of the optimal abatement technologies over which the red surface is illustrated. It is seen that the maximum value of $d\boldsymbol{E}(\theta)/d\theta$ defined over the feasible region of β_i and β_j is negative, numerically confirming the analytical result in Theorem 3.

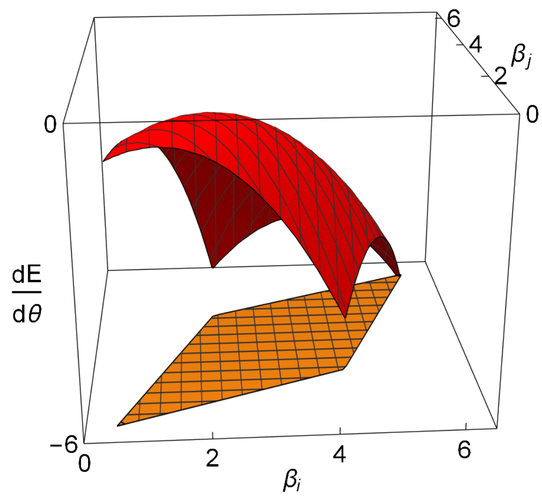


Figure 3. Contorollability of the ambient charge.

5 Optimal Ambient Tax

In this section, we consider how the government determines the ambient tax. Taking the optimal outputs obtained at the second-stage as given, the government constructs the social welfare function,

$$W(\theta) = CS(\theta) + PS(\theta) + T(\theta) - D(\theta) \tag{32}$$

where $CS(\theta)$, $PS(\theta)$, $T(\theta)$ and $D(\theta)$ stand for consumer surplus, producer surplus, tax collected with pollution emission and the damage caused by NPS pollutions, respectively. The price functions (1) and (2) are obtained by maximizing the utility function,

$$U(q_i, q_j, m) = \alpha_i q_i + \alpha_j q_j - \frac{1}{2}\left(q_i^2 + 2\gamma q_i q_j + q_j^2\right) + x$$

subject to the budget constraint, $p_i q_i + p_j q_j + x = I$ where x denotes the numeraire good. Hence, the consumer surplus is

$$CS(\theta) = U(q_i^*, q_j^*, m) - I$$

where the right-hand side is

$$(\alpha_i - p_i^*)\, q_i^* + \left(\alpha_j - p_j^*\right) q_j^* - \frac{1}{2}\left[(q_i^*)^2 + 2\gamma q_i^* q_j^* + \left(q_j^*\right)^2\right].$$

The producer surplus is the sum of the profits gained by the two firms,

$$\begin{aligned} PS(\theta) &= \pi_i^*(\theta) + \pi_j^*(\theta) \\ &= \sum_{k=i}^{j} (p_k^* - c_k)\, q_k^* - 2\theta\left[\phi_i q_i^* + \phi_j q_j^* - \bar{E}\right]. \end{aligned}$$

The tax revenue is

$$T(\theta) = 2\theta\left[\phi_i q_i^* + \phi_j q_j^* - \bar{E}\right]$$

The damage function is assumed to be linear in the total emission,

$$D(\theta) = \delta\left[\phi_i q_i^* + \phi_j q_j^*\right]$$

where $\delta > 0$ measures the marginal damage degree. With substitutions and arrangements, the welfare function is reduced to the form,

$$W(\theta) = (\beta_i - \delta\phi_i)\, q_i^* + \left(\beta_j - \delta\phi_j\right) q_j^* - \frac{1}{2}\left[(q_i^*)^2 + 2\gamma q_i^* q_j^* + \left(q_j^*\right)^2\right]. \tag{33}$$

In the literature of NPS pollution, the government is assumed not to measure the exact levels of firms' productions due to information asymmetry. Here we assume that the government is able to measure their uncertain levels,

$$q_i^* + \varepsilon_i \text{ and } q_j^* + \varepsilon_j$$

where ε_i and ε_j are random variables and assumed to satisfy the followings:

Assumption 4. $\varepsilon_i = \varepsilon_j = \varepsilon$, $E(\varepsilon) = 0$ and $Var(\varepsilon) = \sigma^2$.

The random welfare function has the form,[4]

$$\begin{aligned} W_\varepsilon(\theta) &= (\beta_i - \delta\phi_i)\,(q_i^* + \varepsilon) + \left(\beta_j - \delta\phi_j\right)\left(q_j^* + \varepsilon\right) \\ &-\frac{1}{2}\left[(q_i^* + \varepsilon)^2 + 2\gamma\,(q_i^* + \varepsilon)\left(q_j^* + \varepsilon\right) + \left(q_j^* + \varepsilon\right)^2\right]. \end{aligned}$$

The right-hand side is rewritten as

$$W(\theta) + \left[\left(\beta_i + \beta_j\right) - \delta(\phi_i + \phi_j)\right]\varepsilon - (1+\gamma)\left(q_i^* + q_j^*\right)\varepsilon - (1+\gamma)\,\varepsilon^2. \tag{34}$$

[4]Since the total amount of pollution is

$$\phi_i(q_i^* + \varepsilon) + \phi_j(q_j^* + \varepsilon) = \phi_i q_i^* + \phi_j q_j^* + \left(\phi_i + \phi_j\right)\varepsilon,$$

the government can observe the total amount with a random error $\left(\phi_i + \phi_j\right)\varepsilon$.

Notice that $W_\varepsilon(\theta) = W(\theta)$ if the variance is zero (that is, perfect knowledge on the outputs). The expectation of $W_\varepsilon(\theta)$ is

$$E\left[W_\varepsilon(\theta)\right] = W(\theta) - (1+\gamma)\,\sigma^2 \tag{35}$$

and the variance is

$$Var\left[W_\varepsilon(\theta)\right] = E\left[\left(W_\varepsilon(\theta) - E\left[W_\varepsilon(\theta)\right]\right)^2\right] \tag{36}$$

where the right-hand side is expanded as

$$\left[\mu - (1+\gamma)\left(q_i^* + q_j^*\right)\right]^2 E(\varepsilon^2) + (1+\gamma)^2\left[E\left(\varepsilon^4\right) - \sigma^4\right]$$
$$-2(1+\gamma)\left[\mu - (1+\gamma)\left(q_i^* + q_j^*\right)\right] E\left(\varepsilon^3\right)$$

with $\mu = (\beta_i + \beta_j) - \delta(\phi_i + \phi_j)$.

For given ϕ_i and ϕ_j, the government considers the problem of choosing θ with which the welfare expectation is maximized and the welfare variance is minimized. For this purpose, it defines the composite welfare function,

$$\omega(\theta) = E\left[W_\varepsilon(\theta)\right] - \alpha Var\left[W_\varepsilon(\theta)\right] \tag{37}$$

where $\alpha \geq 0$ denotes the subjective importance of the variance in comparison to expectation. The first-order condition for interior optimum reads

$$\left(\frac{dW(\theta)}{dq_i}\frac{dq_i^*}{d\theta} + \frac{dW(\theta)}{dq_j}\frac{dq_j^*}{d\theta}\right) - \alpha\frac{dVar\left[W_\varepsilon(\theta)\right]}{d\theta} = 0 \tag{38}$$

where the derivative of $Var\left[W_\varepsilon(\theta)\right]$ is simplified as

$$-2\left[(1+\gamma)\sigma^2\left(\mu - (1+\gamma)\left(q_i^* + q_j^*\right)\right) - (1+\gamma)^2 E\left(\varepsilon^3\right)\right]\left(\frac{dq_i^*}{d\theta} + \frac{dq_j^*}{d\theta}\right).$$

The first-order condition (38) can be rewritten

$$\left\{\frac{dW(\theta)}{dq_i} + 2\alpha\left[(1+\gamma)\sigma^2\left(\mu - (1+\gamma)\left(q_i^* + q_j^*\right)\right) - (1+\gamma)^2 E\left(\varepsilon^3\right)\right]\right\}\frac{dq_i^*}{d\theta} +$$

$$\left\{\frac{dW(\theta)}{dq_j} + 2\alpha\left[(1+\gamma)\sigma^2\left(\mu - (1+\gamma)\left(q_i^* + q_j^*\right)\right) - (1+\gamma)^2 E\left(\varepsilon^3\right)\right]\right\}\frac{dq_j^*}{d\theta} = 0.$$

Solving it for θ yields the best reply of the government, $\theta^e\left(\phi_i, \phi_j\right)$ that has the form,

$$\frac{(k_i + k_0)\,\phi_i + (k_j + k_0)\,\phi_j + k_1\left[\phi_i^2 - \gamma\phi_i\phi_j + \phi_j^2 + k_2\left(\phi_i + \phi_j\right)^2\right]}{2\gamma^3\phi_i\phi_j + (4 - 3\gamma^2)\left(\phi_i^2 + \phi_j^2\right) + 2\alpha(2-\gamma)^2(1+\gamma)^2\sigma^2\left(\phi_i + \phi_j\right)^2} \tag{39}$$

where

$$
\begin{aligned}
k_i &= 4\gamma\beta_j - (4+\gamma^2)\beta_i, \\
k_j &= 4\gamma\beta_i - (4+\gamma^2)\beta_j, \\
k_0 &= 2\alpha(2-\gamma)^2(1+\gamma)\left[(2+\gamma)E(\varepsilon^3) - \sigma^2\left(\beta_i+\beta_j\right)\right], \\
k_1 &= 2\delta(2-\gamma)(2+\gamma), \\
k_2 &= \alpha\sigma^2(2-\gamma)(1+\gamma).
\end{aligned}
$$

6 Nash Equilibrium

This section is divided into two subsections. In the first subsection, Nash solutions are obtained and in the second subsection, the comparative statics is considered.

6.1 Determination of Nash Solutions

The firms' best reply functions are $\phi_i^e(\theta)$ and $\phi_j^e(\theta)$ in (14) where the ambient tax is given. On the other hand, $\theta^e(\phi_i,\phi_j)$ in (39) is the government's best reply function, taking the technology levels as given. Solving these equations represents the Nash solutions of the ambient charge and the abatement technology. However, $\theta^e(\phi_i,\phi_j)$ has a complicated form that prevents from gaining a general solution. To proceed, at the expense of the generality, we, first, add an assumption on the government uncertainty. As can be seen in the definition of k_0, $E(\varepsilon^3)$ is an uncertainty effect on the government decision on the ambient charge, and its value depends on the form of the density function of the random variable. It is assumed that $E\left(\varepsilon^3\right)=0$ as it holds for any distribution being symmetric around zero. As can be seen in (39), a non-zero $E\left(\varepsilon^3\right)$ just shifts the government's best reply up or down and thus does not change the determination of the Nash solution drastically. Second, we specify the values of some parameters. These simplifications are summarized:

Assumption 5. $\alpha = 1/4,\ \gamma = 3/5,\ \delta = 1,\ \sigma = 1$ and $E\left(\varepsilon^3\right)=0$.

Under Assumption 5, the form of the government's best reply is simplified as

$$
\frac{182\left(3\phi_i^2+\phi_i\phi_j+3\phi_j^2\right)+5\left(8\beta_j-57\beta_i\right)\phi_i+5\left(8\beta_i-57\beta_j\right)\phi_j}{261\phi_i^2+262\phi_i\phi_j+261\phi_j^2}. \tag{40}
$$

Substituting $\phi_i^e(\theta)$ and $\phi_j^e(\theta)$ into $\theta^e(\phi_i,\phi_j)$ and subtracting it from θ present an equation in θ,

$$
\theta - \theta^e\left[\phi_i^e(\theta),\phi_j^e(\theta)\right]=0. \tag{41}
$$

Solving this equation presents a Nash solution of the ambient charge tax. Using the form of (40), we can rewrite the left-hand side of (41),

$$\theta - \theta^e\left[\phi_i^e(\theta), \phi_j^e(\theta)\right] = \frac{P\varphi(\theta)}{261\phi_i^2 + 262\phi_i\phi_j + 261\phi_j^2}$$

where

$$P = \frac{182}{\left(3014284 - 1820000\theta^2 + 184525\theta^4\right)^2},$$

$$\varphi(\theta) = A_7\theta^7 + A_6\theta^6 + A_5\theta^5 + A_4\theta^4 + A_3\theta^3 + A_2\theta^2 + A_1\theta + A_0 \tag{42}$$

and the coefficient forms are provided in Appendix 5.

Taking $\beta_j = 2$, we determine the Nash solution of θ in Figure 4. The grey region is defined for $\theta \geq \bar{\theta}(\gamma)$ and eliminated for further consideration because Assumption 3 is violated there. Three different colored regions have the following constraints,

$$0 \leq q_i^e \text{ and } 0 \leq q_j^e \text{ in the yellow region,}$$

$$0 \leq \phi_i^e \leq 1 \text{ in the union of the yellow and green regions,}$$

$$0 \leq \phi_j^e \leq 1 \text{ in the union of the yellow, green, and light green regions.}$$

In the yellow region without boundaries, the optimal abatement technology is positive and less than unity, and the optimal outputs are positive,

$$0 < \phi_i^e < 1,\ 0 < \phi_j^e < 1,\ 0 < q_i^e \text{ and } 0 < q_j^e. \tag{43}$$

The black curve describes the $\varphi(\theta) = 0$ locus. Hence, the optimal ambient tax rate takes place along the black curve within the yellow region. This segment starts at N_0 and ends at N_4 where the coordinates are

$$\beta_i^{N_0} \simeq 1.054 \text{ and } \theta^{N_0} \simeq 0.574 \text{ at point } N_0$$

and

$$\beta_i^{N_4} = 3.2 \text{ and } \theta^{N_4} = 0 \text{ at point } N_4.$$

The optimal ambient tax is uniquely determined for any $\beta_i \in (\beta_i^{N_0}, \beta_i^{N_4})$. With this optimal value of θ, the firms select the optimal abatement technologies through equations in (14) and then the optimal outputs through equations in

(19).

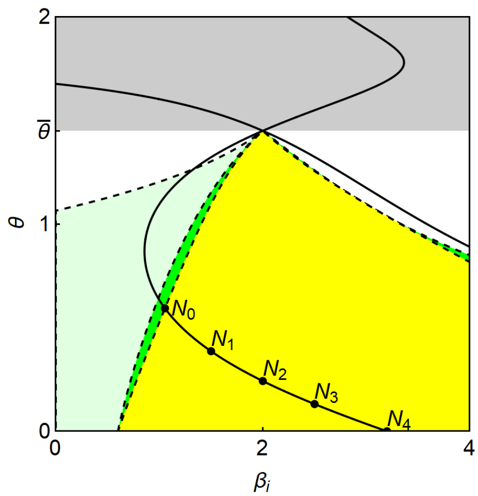


Figure 4. Nash solutions of θ for $\beta_j = 2$

We numerically confirm the optimal solutions. To this end, in addition to N_0 and N_4, we select three points $N_1,\ N_2,\ N_3$ where the values of $\beta_i^{N_k}$ at points N_k are

$$\beta_i^{N_1} = 1.5,\ \ \beta_i^{N_2} = 2,\ \beta_i^{N_3} = 2.5,$$

and calculate the corresponding solutions for $(\beta_i, \beta_j) = (\beta_i^{N_k}, 2)$ with $k = 0, 1, 2, 3, 4$. The solutions are summarized in Table 2 where

$$Q^e = q_i^e + q_j^e \text{ and } \boldsymbol{E} = \sum_k \phi_k^e q_k^e$$

β_i	θ^e	ϕ_i^e	ϕ_j^e	q_i^e	q_j^e	Q^e	$\boldsymbol{E}$	π_i^e	π_j^e
1.05	0.594	0.979	0.714	0.0	0.788	0.788	0.563	0.259	1.133
1.5	0.387	0.915	0.833	0.353	0.733	1.086	0.934	0.268	0.771
2.0	0.243	0.904	0.904	0.685	0.685	1.369	1.239	0.552	0.552
2.5	0.132	0.927	0.953	0.998	0.638	1.636	1.532	1.041	0.415
3.2	0.0	1.0	1.0	1.429	0.571	2.0	2.0	2.041	0.327

Table 2. Summary of the numerical results

Since the optimal values of θ are less than unity and the marginal damage δ is taken to be unity in Assumption 5, the optimal ambinet tax in our model is less than the Pigouvian tax and the difference between θ^e and δ gets larger as the market size becomes larger. Notice that optimal solutions of ϕ_k^e and q_k^e for $k = i, j$ satisfy the inequality conditions in (43). Fixing the market size of firm j at $\beta_j = 2$, firm i increases its market size in four steps from $\beta_i^{N_0}$ to $\beta_i^{N_4}$. At point N_0 where the zero-production curve of firm i intersects the $\varphi(\theta) = 0$ curve, firm i produces no output, and the market is monopolized by firm j.

At point N_2 with $\beta_i = 2$, the two firms are identical, taking the same level of the ambient technology and producing the same amount of output. Along the segment between N_0 and N_2, firm j has a larger market size (i.e., a dominant firm) produces more output and uses a more efficient abatement technology. On the other hand, along the segment between N_2 and N_4, firm i becomes a dominant firm, making larger production and adopting a better technology. At the point N_4, no ambient tax rate is imposed and thus, as a natural consequence, the firms fully discharge emissions.

6.2 Comparative Statics

We can confirm the ambient charge effects on the various variables. Observing Table 2 from the bottom raw to the top raw, we find various reactions of the firms when the ambient tax increases from zero. Firm i produces less output and decreases its profit while it first uses a better abatement technology but later switches it to a worse one. On the other hand, firm j produces more output, takes a better technology and increases its profit. Hence, for firm i

$$\frac{\Delta q_i^e}{\Delta \theta} < 0, \; \frac{\Delta \phi_i^e}{\Delta \theta} \lesseqgtr 0 \text{ and } \frac{\Delta \pi_i^e}{\Delta \theta} < 0$$

where Δ denotes a change in a variable and $\Delta x / \Delta y > 0$ (respectively, < 0) menas that the direction of a change in y is the same as (respectively, opposite to) the direction of a change in x. For firm j,

$$\frac{\Delta q_j^e}{\Delta \theta} > 0, \; \frac{\Delta \phi_j^e}{\Delta \theta} < 0 \text{ and } \frac{\Delta \pi_j^e}{\Delta \theta} > 0.$$

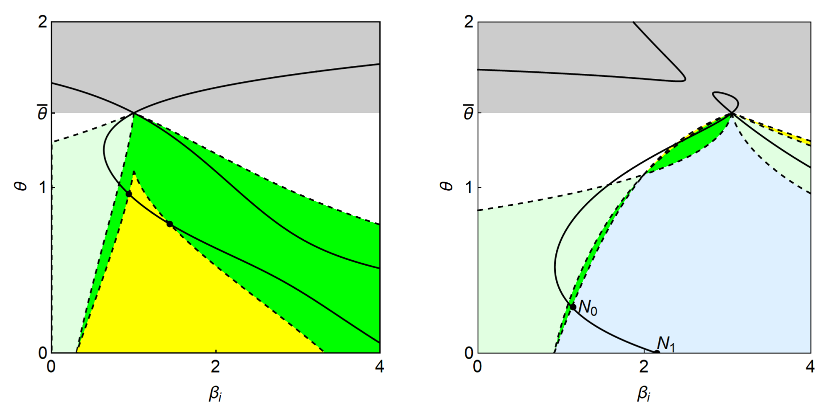
Although the directions of output change for the two firms are opposite, the total amount of output and the total concentration of pollution decline,

$$\frac{\Delta Q^e}{\Delta \theta} < 0 \text{ and } \frac{\Delta \boldsymbol{E}}{\Delta \theta} < 0$$

where the last result is analytically shown in Theorem 3.

We now consider the effects caused by changes in the market size of firm j. In Figure 5(A) in which the value of β_j is dropped to 1 from 2, it is observed first that the triangle-shaped yellow region shrinks and second that the black curve of $\varphi(\theta) = 0$ shifts upward. In consequence, the feasible black curve located in the yellow region is a segment between the two black dots. Hence, the feasible interval of β_j and the corresponding interval of the abatement tax become smaller and the optimal value of the ambient tax is determined to be higher. On the other hand, in Figure 5(B) in which the value of β_j rises to 3.05 from 2, the shape of the feasible region is rather distorted because the size correlation among the three sets are ambiguous. It is seen that some segment of the black curve is in the upper green region that is outside the light green region. Some other segment is in the upper light green region that is outside the green region. Along with those segments of the black curve, the ambient tax can be determined, however, the corresponding ϕ_k^e may be negative or greater than unity. The feasible region of (β_i, θ) in which all inequalities in (43) holds is colored in light blue. Only the solution along the segment between N_0 and N_1 corresponds to Nash solutions of θ. It is seen that the optimal ambient tax decreases due to the larger market size of firm j. The negative market size effect for firm i is already observed in the first two columns of Table 2. Comparing Figure 4 with Figures 5(A) and 5(B), we summarize the main results:

Summary 1 *(1) Under Assumption 5, the optimal ambient tax rate exists as a solution of the two-stage game and is less than the Piguvian tax where* $\delta = 1$. *(2) The market size effect is negative since the larger market size induces decline of the optimal ambient tax. (3) The higher the ambient tax, the lower the total amounts of output and pollution.*



(A) $\beta_j = 1$ (B) $\beta_j = 3.05$

Figure 5. Effects of changing the market size, β_j, of firm j

Let us consider the effects caused by a change in α (*i.e.*, the coefficient of $Var\,[W_\varepsilon(\theta)]$) in the composite welfare function in (37)). In Figure 6, for the benchmark case with Assumption 5 and $\beta_j = 2$, the $\varphi(\theta) = 0$ curve is illustrated in dotted black and so are the curves of $q_i^e(\theta) = 0$ and $q_j^e(\theta) = 0$ in dotted red, respectively. The value of α is increased to 1 from 1/4. The $\varphi(\theta) = 0$ curves with the new α are depicted as solid black curves while the change in α does not affect the zero-production curves and the size of the yellow region. The dotted and solid black curves in the yellow regions are almost the same, although slightly lower departure of the black curve from the dotted black curve is observed in the neighborhood of the crossing point with the positive-sloping dotted red curve. It can be mentioned that an increase in α decreases the optimal value of the ambient charge tax rate. Hence, a change in α positively affects the Nash values of the ambient tax and other variables but only in limited amounts.

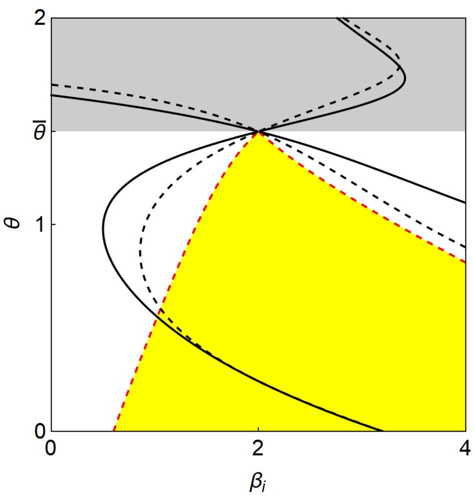


Figure 6. Effects of α when α increases to 1 from 1/4

We now turn attention to the effects caused by a change in the degree of production differentiation when the value of γ rises to 4/5 from 3/5. All inequalities in (43) hold in the feasible region. The feasible regions under Assumption 5 is a union of the yellow and orange regions in Figure 7. The increased value of γ shrinks the yellow region to the orange region, and enlarges the gray region. The segment of the black curve in the feasible region shifts upward. Therefore, a larger value of γ gives rise to a higher ambient charge tax, which is still less than the Pigouvian tax rate. A difference between the optimal ambient tax rate and the Pigouvian tax rate becomes smaller as γ increases.[5]

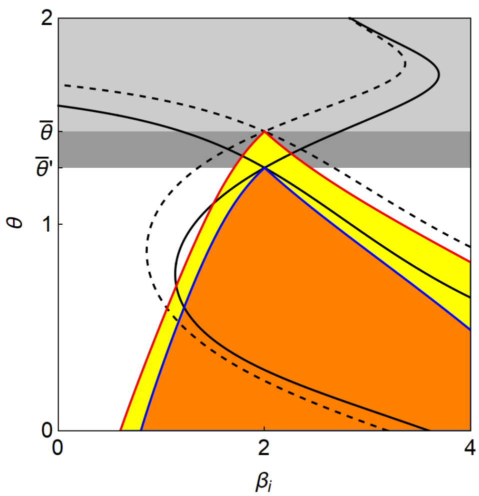


Figure 7. Effects of γ when γ increases to 4/5 from 3/5

Let us next consider how the optimal ambient tax rate reacts when the value of the damage coefficient, δ, changes. We consider two cases, δ is, first, increased to 1.5 from 1 and then to 1.75. In Figure 8, the yellow region is independent of δ and thus remains the same. However, increasing the value of δ shifts the segment of the black curve upward. With the first increment of δ, the feasible segment of $\varphi(\theta) = 0$ is transferred to the solid curve from the dotted curve. In consequence, we observe that the optimal value of θ increases, and the feasible region of β_i is extended. Hence, larger marginal damage induces a larger value of the ambient tax rate. Further, the negative-sloping segment implies that the larger market size makes the optimal value of the ambient tax rate smaller. Those responses are the same as the results obtained shown in Figures 6 and 7 when α and γ are changed. Before proceeding to the second increment, we sum up the results obtained so far:

Summary 2 *A decrease in α, an increase in γ or a small increase in δ shifts the $\varphi(\theta) = 0$ curves upward and increases the optimal value of the ambient charge tax rate.*

The second increment of δ causes a qualitatively different change to the shape of the $\varphi(\theta) = 0$ curve, which is illustrated by the red-green curve in Figure 8. The $\varphi(\theta) = 0$ curve becomes a closed curve and thus has positive sloping parts, as illustrated in red. As a result, solving $\varphi(\theta) = 0$ for θ yields multiple (actually, two) solutions for $\beta_i \in (\beta_i^a, \beta_i^b)$. We numerically check possible results caused

[5] It was checked that the optimal ambient tax is smaller than unity even when γ=1, the maximum value of γ.

by the larger change of δ.

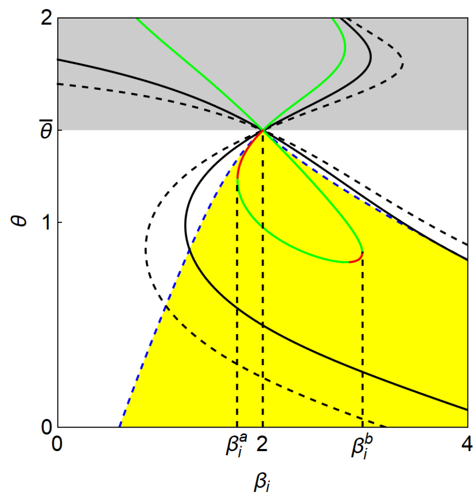


Figure 8. Effets of δ when δ increases to 1.5 and then to 1.75 from 1

First of all, notice that the loci of $\varphi(\theta) = 0$ with different values of δ pass through the common point of $\beta_i = 2$ and $\theta = \bar{\theta}$ as seen in Figure 8. Returning to (15), we see that $A - B = 0$ for $\theta = \bar{\theta}$ and thus the optimal values of the abatement technologies in (15) and the optimal outputs are not defined there. Hence, this point is eliminated in the following considerations. We select several points of β_i out of the interval $[\beta_i^a, \beta_i^b]$ and then calculate the optimal values of the abatement technology and the output. Using the same procedure above, we could summarize the numerical results in a table similar to Table 2. Instead of doing so, we graphically considere them to take care of the discontinuous changes of the variables at the critical point, $(2, \bar{\theta})$. The blue and magenta curves in Figures 9(A) and 9(B) describe the loci of ϕ_i^e and ϕ_j^e and the loci of q_i^e and q_j^e against the selected values of β_i. The solid and dotted curves are obtained along the lower and upper half of the closed red-green curve.

We start to consider firm i's behavior along the lower half of the closed curve. Firm i's optimal choices of ϕ_i and q_i are described by the solid blue curves. The solid blue curve is negative-sloping in Figure 9(A) and positive-sloping in Figure 9(B). Since the optimal ambient tax rate is negatively related with the market size β_i along the lower half, [6] increasing β_i leads to a fall of the ambient tax rate that then induces firm i take a more-abated technology (i,e., $\Delta\phi_i^e < 0$) and produce more output (i.e., $\Delta q_i^e > 0$). The ambient charge effects on ϕ_i^e and q_i^e are obtained,

$$\frac{\Delta\phi_i^e}{\Delta\beta_i} = \frac{\Delta\phi_i^e}{\Delta\theta}\underset{(-)}{\frac{\Delta\theta^e}{\Delta\beta_i}} < 0 \Longrightarrow \frac{\Delta\phi_i^e}{\Delta\theta} > 0 \text{ and } \frac{\Delta q_i^e}{\Delta\beta_i} = \frac{\Delta q_i^e}{\Delta\theta}\underset{(-)}{\frac{\Delta\theta^e}{\Delta\beta_i}} > 0 \Longrightarrow \frac{\Delta q_i^e}{\Delta\theta} < 0.$$

[6] We do not select β_i from the red part at the lower-left only for analytical simplicity.

Firm j's optimal behavior is desribed by the solid magenta curves. The solid magenta curve is positive-sloping in Figure 9(A) and negative-sloping in Figure 9(B). $\Delta\theta^e/\Delta\beta_i < 0$ is already checked. Hence, increasing β_i leads to a fall of the ambient tax rate that then induces firm j take a less-abated technology (i.e., $\Delta\phi_i^e > 0$) and produces less output (i.e., $\Delta q_j^e < 0$). The ambient charge effects on ϕ_j^e and q_j^e are also obtained,

$$\frac{\Delta\phi_j^e}{\Delta\beta_i} = \frac{\Delta\phi_j^e}{\Delta\theta}\underset{(-)}{\frac{\Delta\theta^e}{\Delta\beta_i}} > 0 \Longrightarrow \frac{\Delta\phi_j^e}{\Delta\theta} < 0 \text{ and } \frac{\Delta q_j^e}{\Delta\beta_i} = \frac{\Delta q_j^e}{\Delta\theta}\underset{(-)}{\frac{\Delta\theta^e}{\Delta\beta_i}} < 0 \Longrightarrow \frac{\Delta q_j^e}{\Delta\theta} > 0.$$

Those results allow us to state the following:

Summary 3 *When the value of β_i increases from β_i^a to β_i^b, the ambient tax decreases along the lower half of the $\varphi(\theta) = 0$ closed curve. Accordingly, firm i adopts the more-abated techonology and produces more output whereas firm j adopts the less-abated techonology and produces less ouput,*

$$\frac{\Delta\phi_i^e}{\Delta\theta} > 0 \textit{ and } \frac{\Delta q_i^e}{\Delta\theta} < 0, \textit{ and } \frac{\Delta\phi_j^e}{\Delta\theta} < 0 \textit{ and } \frac{\Delta q_j^e}{\Delta\theta} > 0.$$

We now turn attention to the firms's behavior along the upper half of the closed curve. In Figure 8, it is seen that the red part is defined for $\beta_i^a \le \beta_i < 2$ and positive-sloping (i.e., $\Delta\theta^e/\Delta\beta_i > 0$), and the remaining green part is defined for $2 < \beta_i \le \beta_i^b$ and negative-sloping (i.e., $\Delta\theta^e/\Delta\beta_i < 0$). The dotted blue curve is positive-sloping in Figure 9(A), negative-sloping in Figure 9(B) and discontinuous at $\beta_i = 2$. When the market size becomes larger, the ambient tax rate also increases. Accordingly, firm i takes a less-abated technology, getting closer to full-charge and produces less output for $\beta_i < 2$ but suddenly switch it to a high-efficient technology and produces a larger amount of output just when β_i becomes larger than 2. Then, a less-abated technogoy is used and more output is produced as β_i further increases to β_i^b. Hence, firm i's responses are summarized as follows,

$$\frac{\Delta\phi_i^e}{\Delta\theta} > 0 \text{ and } \frac{\Delta q_i^e}{\Delta\theta} < 0 \text{ for } \beta_i^a \le \beta_i < 2$$

and

$$\frac{\Delta\phi_i^e}{\Delta\theta} < 0 \text{ and } \frac{\Delta q_i^e}{\Delta\theta} < 0 \text{ for } 2 < \beta_i \le \beta_i^b.$$

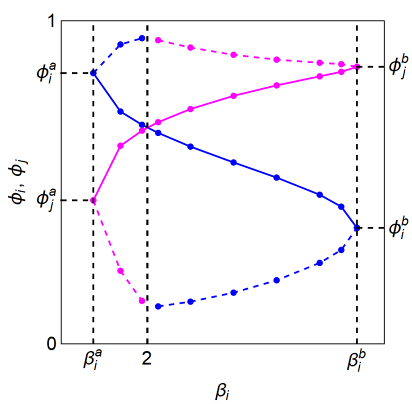
The dotted magenta curve is negative-sloping in Figure 9(A), positive-sloping in Figure 9(B) and becomes discontinuous at $\beta_i = 2$. When the ambient tax rate increases, firm j adopts a more-abated technology and produces more output for $\beta_i < 2$. It drastically changes its decision policy just when β_i becomes larger than 2, actually, it jumps to an almost full-charge technology and decreases its output close to zero. The technology level is getting better and output level is gettling larger as β_i approaches to β_i^b.

$$\frac{\Delta\phi_j^e}{\Delta\theta} < 0 \text{ and } \frac{\Delta q_j^e}{\Delta\theta} > 0 \text{ for } \beta_i^a \le \beta_i < 2$$

and

$$\frac{\Delta\phi_j^e}{\Delta\theta} > 0 \text{ and } \frac{\Delta q_j^e}{\Delta\theta} < 0 \text{ for } 2 < \beta_i \leq \beta_i^b.$$

Summary 4 *When the value of β_i increases from β_i^a to β_i^b, the ambient charge tax rate increases for $\beta_i < 2$ and decreases for $\beta_i > 2$ along the upper half of the $\varphi(\theta) = 0$ closed curve. Firms' optimal choices of the abatement technolgy and output production are drastically changed when the government changes its tax policy from a tax increase to a tax decrease.*



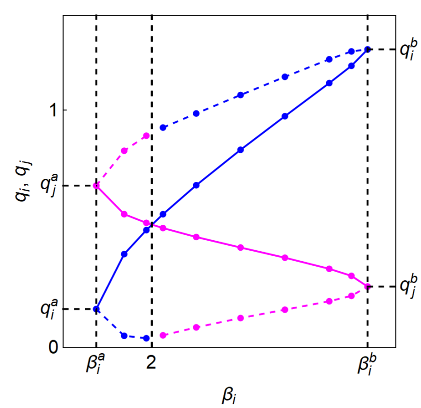


Figure 9. Numerical results when $\delta = 1.75$

Taking β_i given, we have two distinct solutions for θ when the $\varphi(\theta) = 0$ curve is closed, and need a selection criterion to choose one solution. It is a decision problem for the profit-maximizing firms. Hence, the profit amount could be a possible criterion. Figure 10, in which the graphical properties (i.e., meanings of blue or magenta, solid or dotted) are the same, illustrates the profit changes against the selected values of β_i. The dotted curves are located above the solid curves, implying that the profit obtained along the upper half is larger than the profit along the lower half. Therefore, the firms might prefer the choices along the upper half of the closed curve.

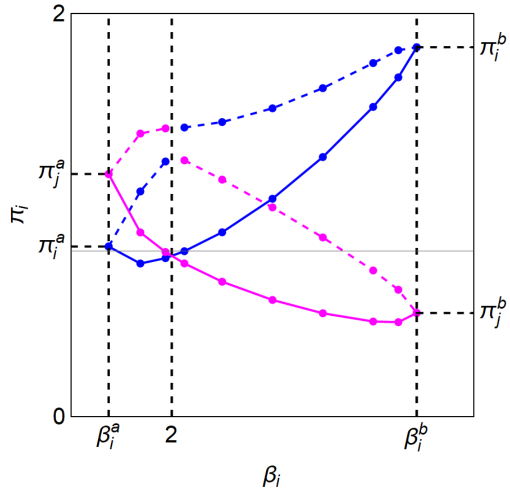


Figure 10. The amounts of profit obtained from two routes.

7 Concluding Remarks

This paper investigates the optimal environmental policy for non-point source pollution under Cournot duopoly competition. The profit-maximizing firms determine their outputs and select the abatement technologies. The government that is unable to monitor the individual emission amounts of the firms constructs a welfare function with uncertainty and determines the ambient tax rate by maximizing the welfare expectation and minimizing the welfare variance. The best replies of the firms depend on the ambient tax rate that the government imposes, and the best reply of the government depends on the levels of the abatement technologies that the firms select. Hence, the optimal ambient charge tax rate and abatement technologies are determined as Nash solutions. We numerically determine the Nash equilibrium since the government's best reply is too complicated to obtainn a general solution. We have shown that the ambient tax rate is set lower than the Pigouvian tax in our imperfect competitive market. We have also shown that the ambient charge effectively controls the total concentration of NPS pollution. We have further shown various individual responses for a change in the ambient charge tax rate, which the government might not observe.

The study reported in this paper can be extended in several directions. With minor modifications, the two-stage game analyzed in this study can be reconstructed as a standard three-stage game in which the output determinations are done at the third stage, the selections of the abatement technology at the second stage, and then the ambient tax rate at the first stage. The linear price and cost functions could be replaced with nonlinear functional forms. Introducing an isoelastic price function and corresponding dynamic extensions are interesting research topics.

Notice that all calculations in the following Appendices are done with Mathematica, version 12.1.

Appendix 1

In this Appendix, the derivatives of ϕ_k^e of (20) are reproduced,

$$\frac{d\phi_k^e(\theta)}{d\theta} = 4\left(\frac{4-\gamma^2}{A^2-B^2}\right)^2 (\ell - m\beta_k + n\beta_{\bar{k}})\,.$$

The derivation presents the following form of the denominator,

$$Den^2 = \left[\left(2(4-\gamma^2)(2-\gamma) - (4+2\gamma-\gamma^2)\theta^2\right)\left(2(4-\gamma^2)(2+\gamma) - (4-2\gamma-\gamma^2)\theta^2\right)\right]^2.$$

Expanding $A^2 - B^2$ gives rise to the form of

$$A^2 - B^2 = (4-\gamma^2)Den$$

that is rewritten as

$$Den = \frac{A^2-B^2}{4-\gamma^2}.$$

The coefficient of β_k has the form

$$m = m_0 - m_2\theta^2 - m_4\theta^4 + m_6\theta^6$$

where

$$\begin{aligned}
m_0 &= 16(4-\gamma^2)^3,\\
m_2 &= 4(4-\gamma^2)(16-24\gamma^2+3\gamma^4),\\
m_4 &= 4(16-28\gamma^2+3\gamma^4),\\
m_6 &= \left(4+2\gamma-\gamma^2\right)\left(4-2\gamma-\gamma^2\right).
\end{aligned}$$

The coefficient of $\beta_{\bar{k}}$ has the form,

$$n = n_0 + n_2\theta^2 - n_4\theta^4 + n_6\theta^6$$

where

$$\begin{aligned}
n_0 &= 4(4-\gamma^2)^4,\\
n_2 &= 4(4-\gamma^2)^2(20-3\gamma^2),\\
n_4 &= 4(4-\gamma^2)(80-30\gamma^2+3\gamma^4),\\
n_6 &= \left(4+2\gamma-\gamma^2\right)\left(4-2\gamma-\gamma^2\right).
\end{aligned}$$

The derivatives of q_k^e in (22) is reproduced,

$$\frac{dq_k^e(\theta)}{d\theta} = 2\frac{(4-\gamma^2)^4}{(A^2-B^2)^2}\left(-\bar{\ell} + \bar{m}\beta_k - \bar{n}\beta_{\bar{k}}\right)$$

and the coefficient of β_k has the form,

$$\bar{m} = 8\theta \left[\bar{m}_0 - \bar{m}_2\theta^2 + \bar{m}_4\theta^4\right].$$

The full forms of $\bar{m}_0$, $\bar{m}_2$ and $\bar{m}_4$ are

$$\bar{m}_0 = 4(4-\gamma^2)^2(4+\gamma^2),$$
$$\bar{m}_2 = 6(4-\gamma^2)(2-\gamma^2),$$
$$\bar{m}_4 = 16 - 4\gamma^2 + \gamma^4.$$

Appendix 2

In this Appendix, we provide the full forms of the coefficients of equation (27). The coefficient of $\beta_i^2 + \beta_j^2$ is

$$A(\gamma,\theta) = 2\left(4-\gamma^2\right)\left(a_0 + a_2\theta^2 + a_4\theta^4 + a_6\theta^6 + a_8\theta^8 + a_{10}\theta^{10}\right)$$

with

$$a_0 = -16\left(4-\gamma^2\right)^5\left(4+\gamma^2\right),$$
$$a_2 = -96\left(4-\gamma^2\right)^3\gamma^2\left(12-\gamma^2\right),$$
$$a_4 = 8\left(4-\gamma^2\right)^2\left(192 - 64\gamma^2 - 28\gamma^4 - \gamma^6\right),$$
$$a_6 = -16\left(4-\gamma^2\right)\left(2-\gamma^2\right)\left(64 - 16\gamma^2 - \gamma^4\right),$$
$$a_8 = 3\left(16 + 4\gamma^2 - \gamma^4\right)\left(16 - 4\gamma^2 + \gamma^4\right),$$
$$a_{10} = \gamma^2\left(4 + 2\gamma - \gamma^2\right)\left(4 - 2\gamma - \gamma^2\right).$$

Then the coefficient of $\beta_i + \beta_j$ is

$$B(\gamma,\theta) = \left(4-\gamma^2\right)\left[2(2-\gamma)^2(2+\gamma) - \left(4 + 2\gamma - \gamma^2\right)\right]^3\left(b_1\theta + b_3\theta^3\right)$$

with

$$b_1 = 2(2-\gamma)(2+\gamma)^2\left(12 - 2\gamma - \gamma^2\right),$$
$$b_3 = \left(4 - 2\gamma - \gamma^2\right)\left(4 + 2\gamma + \gamma^2\right).$$

Next, the coefficient of $\beta_i\beta_j$ is

$$C(\gamma,\theta) = 2\gamma\left(c_0 + c_2\theta^2 + c_4\theta^4 + c_6\theta^6 + c_8\theta^8 + c_{10}\theta^{10}\right)$$

with

$$c_0 = 128\left(4-\gamma^2\right)^6,$$
$$c_2 = 48\left(4-\gamma^2\right)^4\left(12-\gamma^2\right)\left(4-\gamma^2\right),$$

$$c_4 = -64\left(4-\gamma^2\right)^3\left(32-\gamma^2\right)\left(2-\gamma^2\right),$$
$$c_6 = 8\left(4-\gamma^2\right)^2\left(64+16\gamma^2-12\gamma^4-\gamma^6\right),$$
$$c_8 = 24\left(4-\gamma^2\right)^2\left(16+4\gamma^2-\gamma^4\right),$$
$$c_{10} = \left(4-2\gamma-\gamma^2\right)\left(4+2\gamma-\gamma^2\right)\left(16+4\gamma^2-\gamma^4\right).$$

Lastly, the constant term is

$$D(\gamma,\theta) = d_0 + d_2\theta^2 + d_4\theta^4 + d_6\theta^6 + d_8\theta^8$$

with

$$d_0 = -32\left(4-\gamma^2\right)^8(2-\gamma),$$
$$d_2 = 96\left(4-\gamma^2\right)^6(2-\gamma)\,\gamma\left(8-\gamma^2\right),$$
$$d_4 = 96\left(4-\gamma^2\right)^5\left(4+2\gamma-\gamma^2\right)\left(4-8\gamma+\gamma^3\right),$$
$$d_6 = -8\left(4-\gamma^2\right)^3(2+\gamma)(4+2\gamma-\gamma^2)^2\left(32-40\gamma+5\gamma^3\right),$$

and

$$d_8 = 6\left(4-\gamma^2\right)^2(2+\gamma)(4+2\gamma-\gamma^2)^3\left(4-2\gamma-\gamma^2\right).$$

It can be confirmed that $D(\gamma,\theta)=0$ if $\theta=\bar{\theta}(\gamma)$ and $D(\gamma,\theta)<0$ if $\theta<\bar{\theta}(\gamma)$.

Appendix 3

The quadratic polynomial, $g_j(\beta_j)$ in (31), is reproduced,

$$g_j(\beta_j) = A_j(\gamma,\theta)\,\beta_j^2 + B_j(\gamma,\theta)\,\beta_j + C_j(\gamma,\theta).$$

The coefficient of β_j^2 is

$$A_j(\gamma,\theta) = a_{j0} + a_{j2}\theta^2 + a_{j4}\theta^4 + a_{j6}\theta^6 + a_{j8}\theta^8 + a_{j10}\theta^{10}$$

where

$$a_{j0} = 128\left(4-\gamma^2\right)^6(2+\gamma),$$
$$a_{j2} = 64\left(4-\gamma^2\right)^4(2+\gamma)\left(40+8\gamma-\gamma^3\right),$$
$$a_{j4} = 64\left(4-\gamma^2\right)^3\left(48+120\gamma+8\gamma^2-8\gamma^3+2\gamma^4+\gamma^5\right),$$
$$a_{j6} = 32\left(4-\gamma^2\right)^2\left(-288+144\gamma+56\gamma^2-40\gamma^3-12\gamma^4+2\gamma^5+\gamma^6\right),$$
$$a_{j8} = 8\left(4-\gamma^2\right)\gamma^2\left(4-2\gamma-\gamma^2\right)\left(40-8\gamma^2+\gamma^3\right),$$
$$a_{j10} = -4\gamma^2\left(4+2\gamma-\gamma^2\right)\left(4-2\gamma-\gamma^2\right).$$

The coefficient of β_j is

$$B_j(\gamma,\theta) = -4(4-\gamma^2)b_{j1}b_{j2}b_{j3}$$

with

$$b_{j1} = 2(2+\gamma)(4-\gamma^2)\theta - \left(4-2\gamma-\gamma^2\right)\theta^3,$$
$$b_{j2} = 2(2+\gamma)(4-\gamma^2)(12-2\gamma-\gamma^2) + \left(4+2\gamma-\gamma^2\right)\left(4-2\gamma-\gamma^2\right)\theta^2,$$
$$b_{j3} = 4(2-\gamma)(4-\gamma^2)^2 + 24\left(4-\gamma^2\right)\theta^2 - \left(4+2\gamma-\gamma^2\right)\gamma\theta^4.$$

The constant term is

$$C_j(\gamma,\theta) = -\left(4-\gamma^2\right)^2\left[c_{j0} - 8\left(4-\gamma^2\right)^3 c_{j2}\theta^2 - 4\left(4-\gamma^2\right)^2 c_{j4}\theta^4 - 2\left(4-\gamma^2\right)c_{j6}\theta^6 - c_{j8}\theta^8\right]$$

with

$$c_{j0} = -128\left(4-\gamma^2\right)^5\left(4+\gamma^2\right),$$
$$c_{j2} = 128 + 832\gamma + 192\gamma^2 + 16\gamma^4 + 8\gamma^5 - 2\gamma^6 + \gamma^7,$$
$$c_{j4} = 2304 + 1408\gamma - 1216\gamma^2 - 632\gamma^3 - 352\gamma^4 + 16\gamma^5 + 52\gamma^6 + 6\gamma^7 - 3\gamma^8,$$
$$c_{j6} = 6656 - 728\gamma - 4096\gamma^2 + 728\gamma^3 + 992\gamma^4 - 304\gamma^5 - 48\gamma^6 + 64\gamma^7 + 6\gamma^8 - 3\gamma^9,$$
$$c_{j8} = -\left(4+2\gamma-\gamma^2\right)\left(256 + 512\gamma - 384\gamma^2 - 256\gamma^3 + 176\gamma^4 + 64\gamma^5 - 8\gamma^6 + \gamma^8\right).$$

Appendix 4

We present more precise explanations for each step of the step-by-step approach discussed in Section 4.

First step:

$G\left(\beta_i, \beta_j, \gamma, \theta\right)$ is arranged to be $g_i(\beta_i)$ with the coefficients,

$$A_i\left(\gamma,\theta\right) = A(\gamma,\theta),$$
$$B_i\left(\gamma,\theta\right) = B\left(\gamma,\theta\right) + C\left(\gamma,\theta\right)\beta_j,$$
$$C_i\left(\gamma,\theta\right) = A(\gamma,\theta)\beta_j^2 + B\left(\gamma,\theta\right)\beta_j + D\left(\gamma,\theta\right).$$

We can confirm $A(\gamma,\theta) < 0$. Since $0 < \theta < (4-\gamma^2)/2$ and $0 < \gamma < 1$,

$$A(\gamma,2) = 32\gamma^2(4-\gamma^2)^4(8-\gamma^2)^4 > 0,$$

and

$$A\left(\gamma, \frac{4-\gamma^2}{2}\right) = -\frac{\gamma^4(4-\gamma^2)^6(8-\gamma^2)^3(320-8\gamma^2+14\gamma^4-\gamma^6)}{256} < 0.$$

Then the locus of $A(\gamma,\theta) = 0$ is located between the two loci of $\theta = 2$ and $\theta = (4-\gamma^2)/2$. Under Assumption 3, the feasible θ must satisfy $\theta < \tilde{\theta}(\gamma) <$

$(4-\gamma^2)/2$, implying $A(\gamma,\theta)<0$. $C_i(\gamma,\theta)$ is quadratic in β_j and is equal to $D(\gamma,\theta)<0$ for $\beta_j=0$. Its discriminant is

$$B(\gamma,\theta)^2-4A(\gamma,\theta)D(\gamma,\theta)$$

and is numerically shown to be negative for $0<\gamma<1$ and $0<\theta<2$. Hence $C_i(\gamma,\theta)<0$ for any β_j.

Second step:

The discriminant of $g_i(\beta_i)$ is $B_i(\gamma,\theta)^2-4A_i(\gamma,\theta)C_i(\gamma,\theta)$ that can be factored as

$$\boldsymbol{D}_i(\gamma,\theta)=K(\gamma,\theta)\,g_j(\beta_j)$$

where $K(\gamma,\theta)$ has the form,

$$\left[2(2-\gamma)(2+\gamma)2-\left(4-2\gamma-\gamma^2\right)\theta^2\right]^2\left[2(2-\gamma)^2(2+\gamma)-\left(4+2\gamma-\gamma^2\right)\theta^2\right]^3$$

and it is positive due to Assumption 3, implying $K(\gamma,\theta)>0$. As is seen in (31), $g_j(\beta_j)$ is quadratic in β_j and the full forms of $A_j(\gamma,\theta)$, $B_j(\gamma,\theta)$ and $C_j(\gamma,\theta)$ are presented in the Appendix 3.

Third step:

The discriminant of $g_j(\beta_j)$ is

$$\boldsymbol{D}_j(\gamma,\theta)=64\left(4-\gamma^2\right)^3 d_{j1}d_{j2}d_{j3}A(\gamma,\theta)$$

where

$$d_{j1}=2(2-\gamma)(2+\gamma)^2-\left(4-2\gamma-\gamma^2\right)\theta^2$$
$$d_{j2}=8(2-\gamma)(2+\gamma)^2-\left(16-8\gamma+4\gamma^3+\gamma^4\right)\theta^2$$
$$d_{j3}=4(2-\gamma)^3(2+\gamma)^2+\theta^2\left[24\left(4-\gamma^2\right)-\left(4+2\gamma-\gamma^2\right)\gamma\theta^2\right].$$

Due to the following inequalities for $0<\gamma<1$,

$$\theta<\tilde{\theta}(\gamma)<\sqrt{\frac{8(2+\gamma)(2+\gamma)^2}{16-8\gamma+4\gamma^3+\gamma^4}}<\sqrt{\frac{24\left(4-\gamma^2\right)}{\left(4+2\gamma-\gamma^2\right)\gamma}},$$

we have

$$d_{j1}>0,\ d_{j2}>0 \text{ and } d_{j3}>0.$$

The sign of $\boldsymbol{D}_j(\gamma,\theta)$ is the same as the sign of $A(\gamma,\theta)$. It is already shown that $A(\gamma,\theta)<0$ for $0<\gamma<1$ and $\theta<\tilde{\theta}(\gamma)$. Hence $\boldsymbol{D}_j(\gamma,\theta)<0$, implying $g_j(\beta_j)<0$ that, in turn, implies $\boldsymbol{D}_i(\gamma,\theta)<0$. Since the function $g_i(\beta_i)$ with $A(\gamma,\theta)<0$ is concave and negative for all $\beta_i>0$. The form of $g_i(\beta_i)$ is essentially the same as that of of $G\left(\beta_i,\beta_j,\gamma,\theta\right)$. Therefore, we have

$$G\left(\beta_i,\beta_j,\gamma,\theta\right)<0.$$

Appendix 5

Here we present the coefficient of $\varphi(\theta)$ given in equation (42),

$$A_7 = 156250\left[1085097\left(\beta_i^2+\beta_j^2\right)+1314364\beta_i\beta_j\right],$$

$$A_6 = -625\left[3553326777\left(\beta_i+\beta_j\right)+302730000\left(\beta_i^2+\beta_j^2\right)+3553326777\beta_i\beta_j\right],$$

$$A_5 = -18200\left[-484968484-166541375\left(\beta_i+\beta_j\right)+78916875\left(\beta_i^2+\beta_j^2\right)+32211875\beta_i\beta_j\right],$$

$$A_4 = 4550\left[-3152295146+2661854195\left(\beta_i+\beta_j\right)+247350000\left(\beta_i^2+\beta_j^2\right)+98200000\beta_i\beta_j\right],$$

$$A_3 = 331240\left[-112224112-38538500\left(\beta_i+\beta_j\right)+14079375\left(\beta_i^2+\beta_j^2\right)-6047500\beta_i\beta_j\right],$$

$$A_2 = -165620\left[-364728364+129600835\left(\beta_i+\beta_j\right)+14850000\left(\beta_i^2+\beta_j^2\right)-12550000\beta_i\beta_j\right],$$

$$A_1 = -24114272\left[-1623076-557375\left(\beta_i+\beta_j\right)+185625\left(\beta_i^2+\beta_j^2\right)-156875\beta_i\beta_j\right],$$

$$A_0 = 2^3\cdot 7^7\cdot 13^5\left[-26+5\left(\beta_i+\beta_j\right)\right].$$

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